N. I. AKHIEZER
AND
I. M. GLAZMAN

THEORY OF LINEAR OPERATORS IN HILBERT SPACE

Volume 1

An up-to-date account of developments in an important and rewarding field of basic research in mathematics

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THEORY OF LINEAR OPERATORS IN HILBERT SPACE
is one of the classic textbooks in its field, invaluable for every mathematician and physicist.

THEORY OF LINEAR OPERATORS IN HILBERT SPACE

N. I. Akhiezer and I. M. Glazman

THEORY OF LINEAR OPERATORS IN HILBERT SPACE

(Teoriia lineinykh operatorov v Gil'bertovom prostranstve, Chapters I to V)

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TRANSLATOR'S PREFACE

The present volume comprises the first five chapters of the Russian text of seven chapters and two appendices. It is hoped that it will be a worthwhile addition to the few books on linear operators in English. Present plans call for bringing out the balance of the work in a second volume.

A few words should be said regarding the translation. It was the translator's desire to produce a faithful translation and to convey the intent of the authors in clear English. When conflicts arose, they were resolved in favor of the latter objective. Therefore, a fair degree of freedom was exercised in the translation. Substantial alterations are accompanied by translator's notes.

The translator owes much to Dr. Philip M. Anselone, of the United States Army Mathematics Research Center, University of Wisconsin, who critically read the text, contributed the translator's notes, and did much to polish both the mathematics and style of the text. Thanks go also to Dr. Hans Bueckner of the United States Army Mathematics Research Center, University of Wisconsin, and to Professor Phillip Hartman of The John Hopkins University, for reading critically the entire text. However, for any errors in translation the translator assumes full responsibility.

M.K.N.

AUTHOR'S PREFACE

The present book stems from lectures and papers which were presented by the authors at the Kharkov Mathematics Institute. It is not exhaustive and perhaps should be called "An Introduction to Linear Operators in Hilbert Space." However, the geometry of Hilbert space and the spectral theory of unitary and self-adjoint operators is presented in detail. The book contains some results obtained by Soviet mathematicians.

According to the original plan, we had hoped also to present as applications the theory of Jacobi matrices in connection with the exponential moment problem on the whole axis and the theory of integral equations with Carleman kernel. In any case, the size of the book without these forced us to abandon this intention. It was with special regret that we decided against the presentation of the moment problem since, on one hand, it plays a big role in the development of the theory of operators and, on the other, is a beautiful application for illustrating the theory.

We recommend to the reader that, after the study of this book, he become acquainted with both the moment problem and the theory of the Carleman integral operator.[1]

The book consists of seven chapters of basic text and two appendices. The first appendix is devoted to generalized extensions of symmetric operators. Here results obtained by M. A. Naimark (Moscow) and M. A. Krein (Odessa) have a classical character; these results were placed in an appendix only so that the basic text could be kept comparatively elementary. The second supplement is devoted to the theory of differential operators, which are here an application of the general theory. We do not give a complete list of literature on the theory of operators, but cite only basic papers and recent or little known books. We list a few titles from a general course.[2]

The attentive reader will notice that we occasionally repeat some formulas or proofs. We do this to keep the reader from being exhausted by too much reference to preceding material.

Further, we might have reduced the size of the book somewhat by omitting chapter five, which contains the spectral analysis of completely continuous self-adjoint operators; however, we decided to keep this material as a very natural introduction to the general spectral theory.

[1] Cf. for example, N. I. Akhiezer [2], [3], M. G. Krein and M. A. Krasnoselski [1].
[2] M. H. Stone [3], A. I. Plesner [1], A. I. Plesner and V. H. Rokhlin [1], V. I. Smirnov [1].

The first three chapters, chapter five and chapter six (with the exception of sections 69–73) were written by N. I. Akhiezer and the remainder by I. M. Glazman. However, the final editing of all the material was done jointly. Hence each of us bears responsibility for the whole book.

We consider it our duty to express deep thanks to M. G. Krein for a number of valuable remarks. We are also very grateful to G. I. Drinfeld, M. A. Krasnoselski, V. A. Martchenko and A. Ia. Povzner for remarks and corrections to improve individual parts of the book.

CONTENTS

Chapter I. HILBERT SPACE

1. Linear Spaces 1
2. The Scalar Product 2
3. Some Topological Concepts 4
4. Hilbert Space 5
5. Linear Manifolds and Subspaces 7
6. The Distance from a Point to a Subspace 8
7. Projection of a Vector on a Subspace 10
8. Orthogonalization of a Sequence of Vectors 13
9. Complete Orthonormal Systems 17
10. The Space L^2 21
11. Complete Orthonormal Systems in L^2 24
12. The Space L_a^2 27
13. The Space of Almost Periodic Functions 29

Chapter II. LINEAR FUNCTIONALS AND BOUNDED LINEAR OPERATORS 30

14. Point Functions 30
15. Linear Functionals 32
16. The Theorem of F. Riesz 33
17. A Criterion for the Closure in H of a Given System of Vectors 35
18. A Lemma Concerning Convex Functionals 36
19. Bounded Linear Operators 39
20. Bilinear Functionals 40
21. The General Form of a Bilinear Functional 42
22. Adjoint Operators 43
23. Weak Convergence in H 44
24. Weak Compactness 46

Chapter II (continued)

25. A Criterion for the Boundedness of an Operator 48
26. Linear Operators in a Separable Space 48
27. Completely Continuous Operators 56
28. A Criterion for Complete Continuity of an Operator 57
29. Sequences of Bounded Linear Operators 61

Chapter III. PROJECTION OPERATORS AND UNITARY OPERATORS 63

30. Definition of a Projection Operator 63
31. Properties of Projection Operators 63
32. Operations Involving Projection Operators 65
33. Monotone Sequences of Projection Operators 68
34. The Aperture of Two Linear Manifolds 69
35. Unitary Operators 72
36. Isometric Operators 73
37. The Fourier-Plancherel Operator 74

Chapter IV. GENERAL CONCEPTS AND PROPOSITIONS IN THE THEORY OF LINEAR OPERATORS 78

38. Closed Operators 78
39. The General Definition of an Adjoint Operator 79
40. Eigenvectors, Invariant Subspaces and Reducibility of Linear Operators 81
41. Symmetric Operators 85
42. More about Isometric and Unitary Operators 87
43. The Concept of the Spectrum (Particularly of a Self-Adjoint Operator) 88
44. The Resolvent 91
45. Conjugation Operators 94
46. The Graph of an Operator 95
47. Matrix Representations of Unbounded Symmetric Operators 98

Chapter IV (continued)

48. The Operation of Multiplication by the Independent Variable 103
49. A Differential Operator 106
50. The Inversion of Singular Integrals 114

Chapter V. SPECTRAL ANALYSIS OF COMPLETELY CONTINUOUS OPERATORS 117

51. A Lemma 117
52. Properties of the Eigenvalues of Completely Continuous Operators in R 118
53. Further Properties of Completely Continuous Operators 122
54. The Existence Theorem for Eigenvectors of Completely Self-Adjoint Operators 124
55. The Spectrum of a Completely Continuous Self-Adjoint Operator in R 127
56. Completely Continuous Normal Operators 129
57. Applications to the Theory of Almost Periodic Functions 132

BIBLIOGRAPHY 139

INDEX 145

Chapter I

HILBERT SPACE

1. Linear Spaces

A set R of elements f, g, h, . . . , (also called points or vectors) forms a *linear space* if

(a) there is an operation, called addition and denoted by the symbol $+$, with respect to which R is an abelian group (the zero element of this group is denoted by 0);

(b) multiplication of elements of R by (real or complex) numbers $\alpha, \beta, \gamma, \ldots$ is defined such that

$$\alpha(f+g)=\alpha f+\alpha g,$$
$$(\alpha+\beta)f=\alpha f+\beta f,$$
$$\alpha(\beta f)=(\alpha\beta)f,$$
$$1\cdot f=f, \quad 0\cdot f=0.$$

Elements $f_1, f_2, \ldots, f_n$ in R are *linearly independent* if the relation

$$\alpha_1 f_1+\alpha_2 f_2+\ldots+\alpha_n f_n=0 \tag{1}$$

holds only in the trivial case with $\alpha_1=\alpha_2=\ldots=\alpha_n=0$; otherwise $f_1, f_2, \ldots, f_n$ are *linearly dependent*. The left member of equation (1) is called a *linear combination* of the elements $f_1, f_2, \ldots, f_n$. Thus, linear independence of $f_1, f_2, \ldots, f_n$ means that every nontrivial linear combination of these elements is different from zero. If one of the elements $f_1, f_2, \ldots, f_n$ is equal to zero, then these elements are evidently linearly dependent. If, for example, $f_1=0$, then we obtain the nontrivial relation (1) by taking

$$\alpha_1=1, \alpha_2=\alpha_3=\ldots=\alpha_n=0.$$

A linear space R is *finite dimensional* and, moreover, *n-dimensional* if R contains n linearly independent elements and if any $n+1$ elements of R are linearly dependent. Finite dimensional linear spaces are studied in linear algebra. If a linear space has arbitrarily many linearly independent elements, then it is *infinite dimensional.*

2. The Scalar Product

A linear space R is *metrizable* if for each pair of elements $f, g \in R$ there is a (real or complex) number (f, g) which satisfies the conditions:[1]

(a) $(g,f) = \overline{(f,g)}$,

(b) $(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1(f_1,g) + \alpha_2(f_2,g)$,

(c) $(f,f) \geqq 0$, with equality only for $f = 0$.

The number (f,g) is called the *scalar product*[2] of f and g.

Property (b) expresses the linearity of the scalar product with respect to its first argument. The analogous property with respect to the second argument is

$$(\bar{b}) \qquad (f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1(f,g_1) + \bar{\beta}_2(f,g_2).$$

Property $(\bar{b})$ is derived as follows:

$$(f, \beta_1 g_1 + \beta_2 g_2) = \overline{(\beta_1 g_1 + \beta_2 g_2, f)} = \overline{\beta_1(g_1,f) + \beta_2(g_2,f)} = \\ = \bar{\beta}_1(f,g_1) + \bar{\beta}_2(f,g_2).$$

The positive square root $\sqrt{(f,f)}$ is called the *norm* of the element (vector) f and is denoted by the symbol $\|f\|$. The norm is analogous to the length of a line segment. As with line segments, the norm of a vector is zero if and only if it is the zero vector. In addition, it follows that

$$1^\circ \qquad \|\alpha f\| = |\alpha| \cdot \|f\|.$$

This is shown by using properties (b) and $(\bar{b})$ of the scalar product:

$$(\alpha f, \alpha f) = \alpha(f, \alpha f) = \alpha\bar{\alpha}(f,f) = |\alpha|^2(f,f),$$

from which 1° follows.

We shall prove that for any two vectors f and g,

$$2^\circ \qquad |(f,g)| \leqq \|f\| \cdot \|g\|,$$

with equality if and only if f and g are linearly dependent. We call 2° the Cauchy-Bunyakovski inequality[3], because in the two most important particular cases, about which we shall speak below, it was first used by Cauchy and Bunyakovski.

In the proof of 2°, we may assume that $(f,g) \neq 0$. Letting

$$\theta = \frac{(f,g)}{|(f,g)|},$$

we find that for any real λ,

$$0 \leqq (\bar{\theta} f + \lambda g, \bar{\theta} f + \lambda g) = \lambda^2(g,g) + 2\lambda|(f,g)| + (f,f).$$

[1] A bar over a complex number denotes complex conjugation.

[2] Translator's Note: The phrase *inner product* is also used. Henceforth, we shall call R an *inner product space*.

[3] Translator's Note: This is often called the Schwarz or the Cauchy-Schwarz inequality.

On the right we have a quadratic in λ. For real λ this polynomial is non-negative, which implies that

$$|(f,g)|^2 \leqq (f,f)\cdot(g,g),$$

and this proves 2°. The equality sign will hold only in case the polynomial under consideration has a double root, in other words, only if

$$\bar{\theta} f + \lambda g = 0$$

for some real λ. But this equation implies that the vectors f and g are linearly dependent.

We shall derive one more property of the norm, the inequality,

3° $$\|f+g\| \leqq \|f\| + \|g\|.$$

There is equality if $f=0$ or $g=\lambda f$, $\lambda \geqq 0$. This property is called the *triangle inequality*, by analogy with the inequality for the sides of a triangle in elementary geometry.

In order to prove the triangle inequality, we use the relation

$$\|f+g\|^2 = (f+g,f+g) = (f,f) + (f,g) + (g,f) + (g,g).$$

Hence, by the Cauchy-Bunyakovski inequality

$$\|f+g\|^2 \leqq \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2 = \{\|f\| + \|g\|\}^2$$

which implies that

$$\|f+g\| \leqq \|f\| + \|g\|.$$

For equality, it is necessary that

$$(g,f) = \|f\|\cdot\|g\|.$$

If $f \neq 0$, then, by 2°, it is necessary that

$$g = \lambda f$$

for some λ. From this it is evident that

$$\lambda(f,f) = \|f\|\cdot\|\lambda f\|,$$

whence it also follows that $\lambda \geqq 0$.

An inner product space R becomes a *metric space*, if the distance between two points $f, g \in R$ is defined as

$$D[f,g] = \|f-g\|.$$

It follows from the properties of the norm that the distance function satisfies the usual conditions.[4]

The scalar product yields a definition for the angle between two vectors. However, for what follows, this concept will not be needed. We confine ourselves to the more limited concept of *orthogonality*. Two vectors f and g are *orthogonal* if

$$(f,g) = 0.$$

[4] These conditions are
(a) $D[f,g] = D[g,f] > 0$ (for $f \neq g$),
(b) $D[f,f] = 0$,
(c) $D[f,g] \leqq D[f,h] + D[h,g]$ (triangle inequality).

3. Some Topological Concepts

In the present section we consider some general concepts which are introduced in the study of point sets in an arbitrary metric space. We denote a metric space by E, and speak of the distance $D[f, g]$ between two elements of E. Let us bear in mind that in what will follow we shall consider only the case with E = R and $D[f, g] = \|f - g\|$, i.e., the case with the metric generated by a scalar product.

If f_0 is a fixed element of E, and ρ is a positive number, then the set of all points f for which

$$D[f, f_0] < \rho$$

is called the *sphere* in E with *center* f_0 and *radius* ρ. Such a sphere is a neighborhood, more precisely a ρ-neighborhood of the point f_0.

We say that a sequence of points $f_n \in E$ $(n = 1, 2, 3, \ldots)$ has the *limit point* $f \in E$, and we write

$$f_n \to f \text{ or } \lim_{n\to\infty} f_n = f \tag{1}$$

when

$$\lim_{n\to\infty} D[f_n, f] = 0. \tag{2}$$

It is not difficult to see that (1) implies

$$\lim_{m,\, n\to\infty} D[f_m, f_n] = 0 \tag{3}$$

where m and n tend to infinity independently. In fact, by the triangle inequality,

$$D[f_m, f_n] \leqq D[f_m, f] + D[f_n, f].$$

But the converse is not always correct, i.e., if for the sequence $f_n \in E$ $(n = 1, 2, 3, \ldots)$ relation (3) holds, then there may not exist an element $f \in E$ to which the sequence converges. If (3) is satisfied, then the sequence is called *fundamental*. Thus, a fundamental sequence may or may not converge to an element of the space.

A metric space E is called *complete* if every fundamental sequence in E converges to some element of the space. If a metric space is not complete, then it is possible to complete it by introducing certain new elements. This operation is similar to the introduction of irrational numbers by Cantor's method.

If each neighborhood of $f \in E$ contains infinitely many points of a set M in E, then f is called a limit point of M. If a set contains all its limit points, then it is said to be *closed*. The set consisting of M and its limit points is called the *closure* of M and is denoted by $\bar{M}$.

If the metric space E is the closure of some countable subset of E, then E is said to be *separable*. Thus, in a separable space there exists a countable set N such that, for each point $f \in E$ and each $\varepsilon > 0$, there exists a point $g \in N$ such that

$$D[f,g] < \varepsilon.$$

4. Hilbert Space

A *Hilbert space* H is an infinite dimensional inner product space which is a complete metric space with respect to the metric generated by the inner product. This definition, similar to those in preceding sections, has an axiomatic character. Various concrete linear spaces satisfy the conditions in the definition. Therefore, H is often called an *abstract* Hilbert space, and the concrete spaces mentioned are called *examples* of this abstract space.

One of the important examples of H is the space l^2. The construction of the general theory, to which the present book is devoted, was begun for this particular space by Hilbert in connection with his theory of linear integral equations. The elements of the space l^2 are sequences (of real or complex numbers)

$$f = \{x_n\}_1^\infty, \qquad g = \{y_n\}_1^\infty, \ldots,$$

such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty, \qquad \sum_{n=1}^{\infty} |y_n|^2 < \infty, \ldots$$

The numbers $x_1, x_2, x_3, \ldots$, are called *components* of the vector f or *coordinates* of the point f. The zero vector is the vector with all components zero. The addition of vectors is defined by the formula

$$f + g = \{x_n + y_n\}_1^\infty.$$

The relation

$$\sum_{n=1}^{\infty} |x_n + y_n|^2 < \infty$$

follows from the inequality

$$|x+y|^2 \leqq 2|x|^2 + 2|y|^2.$$

The multiplication of a vector f by a number λ is defined by

$$\lambda f = \{\lambda x_n\}_1^\infty.$$

The scalar product in the space l^2 has the form

$$(f,g) = \sum_{n=1}^{\infty} x_n \bar{y}_n.$$

The series on the right converges absolutely because

$$|xy| \leqq \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2.$$

The inequality

$$|(f,g)| \leqq \|f\| \cdot \|g\|$$

now has the form

$$|\sum_{n=1}^{\infty} x_n \bar{y}_n| \leqq \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \cdot \sqrt{\sum_{n=1}^{\infty} |y_n|^2}$$

and is due to Cauchy.

The space l^2 is separable. A particular countable dense subset of l^2 consists of all vectors with only finitely many nonzero components and with these components rational, i.e., the components are of the form $\xi + i\eta$ where ξ and η are rational numbers.

In addition, the space l^2 is complete. In fact, if the sequence of vectors

$$f^{(k)} = \{x_n^{(k)}\}_{n=1}^{\infty} \qquad (k = 1, 2, 3, \ldots)$$

is fundamental, then each of the sequences of numbers

$$\{x_n^{(k)}\}_{k=1}^{\infty} \qquad (n = 1, 2, 3, \ldots)$$

is fundamental and, hence, converges to some limit x_n $(n = 1, 2, 3, \ldots)$. Now, for each $\varepsilon > 0$ there exists an integer N such that for $r > N, s > N$

$$\sqrt{\sum_{n=1}^{\infty} |x_n^{(r)} - x_n^{(s)}|^2} < \varepsilon.$$

Consequently, for every m,

$$\sqrt{\sum_{n=1}^{m} |x_n^{(r)} - x_n^{(s)}|^2} < \varepsilon.$$

Let s tend to infinity to obtain

$$\sqrt{\sum_{n=1}^{m} |x_n^{(r)} - x_n|^2} \leqq \varepsilon.$$

But, because this is true for every m,

$$\sqrt{\sum_{n=1}^{\infty} |x_n^{(r)} - x_n|^2} \leqq \varepsilon.$$

Hence, it follows that

$$f = \{x_n\}_1^{\infty} \in l^2,$$

and, since $\varepsilon > 0$ is arbitrary,

$$f^{(k)} \to f.$$

Thus, the completeness of the space l^2 is established.

As we demonstrated, the space l^2 is separable. Originally, the requirement of separability was included in the definition of an abstract Hilbert space. However, as time passed it appeared that this requirement was not necessary for a great deal of the theory, and therefore, it is not included in our definition of the space H.

But the requirement of completeness is essential for almost all of our considerations. Therefore, it is included in the definition of H. The appropriate reservation is made in the cases for which this requirement is superfluous.

The space l^2 is infinite dimensional because the *unit vectors*

$$e_1 = \{1, 0, 0, \ldots\},$$
$$e_2 = \{0, 1, 0, \ldots\},$$
$$e_3 = \{0, 0, 1, \ldots\},$$
$$\ldots\ldots\ldots,$$

are linearly independent. The space l^2 is the infinite dimensional analogue of E_m, the (complex) m-dimensional *Euclidean space*, the elements of which are finite sequences

$$f = \{x_n\}_1^m,$$

and most of the theory which we present consists of generalizations to H of well-known facts concerning E_m.

5. Linear Manifolds and Subspaces

One often considers particular subsets of R (and, in particular, of H). Such a subset L is called a *linear manifold* if the hypothesis $f, g \in L$ implies that $\alpha f + \beta g \in L$ for arbitrary numbers α and β. One of the most common methods of obtaining a linear manifold is the construction of a *linear envelope*. The point of departure is a finite or infinite set M of elements of R. Consider the set L of all finite linear combinations

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n$$

of elements $f_1, f_2, \ldots, f_n$ of M. This set L is the smallest linear manifold which contains M. It is called the linear envelope of M or the linear manifold spanned by M. If R is a metric space, then the closure of the linear envelope of a set M is called the *closed linear envelope* of M.

In what follows, closed linear manifolds in H will have a particularly important significance. Each such manifold G is a linear space, metrizable with respect to the scalar product defined in H. Furthermore, G is complete. In fact, every fundamental sequence of elements of G has a limit in H because H is complete, and this limit must belong to G because G is closed. From what has been said, it follows that G itself is a Hilbert space if it contains an infinite number of linearly independent elements; otherwise G is a Euclidean space. Therefore, G is called a *subspace* of the space H.

6. The Distance from a Point to a Subspace

Consider a linear manifold L which is a *proper subset* of H. Choose a point $h \in H$ and let

$$\delta = \inf_{f \in L} \| h - f \|.$$

The question arises as to whether there exists a point $g \in L$ for which

$$\| h - g \| = \delta.$$

In other words, is there a point in L nearest to the point h?[5]

We prove first that there exists at most one point $g \in L$ such that $\delta = \| h - g \|$. Assume that there exist two such points, g' and g''. Since $\frac{1}{2}(g' + g'') \in L$, we have

$$\left\| h - \frac{g' + g''}{2} \right\| \geqq \delta;$$

on the other hand

$$\left\| h - \frac{g' + g''}{2} \right\| \leqq \frac{1}{2}\| h - g' \| + \frac{1}{2}\| h - g'' \| = \delta.$$

Consequently,

$$\left\| h - \frac{g' + g''}{2} \right\| = \delta$$

and therefore

$$\left\| h - \frac{g' + g''}{2} \right\| = \frac{1}{2}\| h - g' \| + \frac{1}{2}\| h - g'' \|.$$

But this is the triangle inequality with the sign of equality. Since

$$h - g' \neq 0$$

we have

$$h - g'' = \lambda(h - g')$$

for some $\lambda \geqq 0$. If $\lambda = 1$, the proof is complete. If $\lambda \neq 1$, then

$$h = \frac{g'' - \lambda g'}{1 - \lambda}$$

so that $h \in L$, which contradicts our assumption. Thus, our assertion is proved.

But, in general, does there exist a point $g \in L$ nearest to the point h? In the most important case, the answer is yes, and the following theorem holds.

THEOREM: *If* G *is a subspace of the space* H *and if*

$$\delta = \inf_{f \in G} \| h - f \|,$$

[5] Translator's Note. The case with $h \in L$ is trivial. Henceforth, the author assumes without saying so that $h \notin L$.

then there exists a vector $g \in G$ (*its uniqueness was proved above*) *for which*

$$\|h - g\| = \delta.$$

Proof: According to the definition of the greatest lower bound, there exists an infinite sequence of vectors $\{g_n\}_1^\infty$, in G, for which

$$\lim_{n\to\infty} \|h - g_n\| = \delta.$$

Now,

$$\left\|h - \frac{g_m + g_n}{2}\right\| \leqq \frac{1}{2}\|h - g_m\| + \frac{1}{2}\|h - g_n\|.$$

Therefore

$$\overline{\lim_{m,\,n\to\infty}} \left\|h - \frac{g_m + g_n}{2}\right\| \leqq \delta,$$

and since

$$\left\|h - \frac{g_m + g_n}{2}\right\| \geqq \delta$$

we have

$$\lim_{m,\,n\to\infty} \left\|h - \frac{g_m + g_n}{2}\right\| = \delta.$$

In the easily proved relation,

$$2\|f'\|^2 + 2\|f''\|^2 = \|f' + f''\|^2 + \|f' - f''\|^2,$$

let

$$f' = h - g_m, \qquad f'' = h - g_n$$

to obtain

$$\|g_n - g_m\|^2 = 2\|h - g_m\|^2 + 2\|h - g_n\|^2 - 4\left\|h - \frac{g_m + g_n}{2}\right\|^2.$$

Therefore,

$$\lim_{m,\,n\to\infty} \|g_n - g_m\| = 0.$$

So the sequence of vectors $\{g_n\}_1^\infty$ converges to some vector $g \in G$. It remains to prove that

$$\|h - g\| = \delta.$$

Now

$$\lim_{n\to\infty} \|g - g_n\| = 0, \qquad \lim_{n\to\infty} \|h - g_n\| = \delta$$

and

$$\|h - g\| \leqq \|h - g_n\| + \|g - g_n\|;$$

consequently

$$\|h - g\| \leqq \delta.$$

But, by the hypothesis of the theorem, $\|h - g\| \geqq \delta$. Thus, the theorem is proved.

7. Projection of a Vector on a Subspace

Let G be a subspace of H. By the preceding section, to each element $h \in H$ there corresponds a unique element $g \in G$ such that

$$\|h - g\| = \inf_{g' \in G} \|h - g'\|. \tag{1}$$

Considering h and g as points, we say that g is the point of the subspace G nearest the point h. If the elements g and h are considered as vectors, then it is said that g is the particular vector of G which deviates least from h. Now, using (1), we show that the vector $h-g$ is orthogonal to the subspace G; i.e., orthogonal to every vector $g' \in G$.

For the proof, we assume that the vector $h-g$ is not orthogonal to every vector $g' \in G$. Let

$$(h - g, g_0) = \sigma \neq 0 \qquad (g_0 \in G).$$

We define the vector

$$g^* = g + \frac{\sigma}{(g_0, g_0)} g_0 \in G.$$

Then

$$\|h - g^*\|^2 = \left(h - g - \frac{\sigma}{(g_0, g_0)} g_0, h - g - \frac{\sigma}{(g_0, g_0)} g_0\right) =$$

$$= \|h - g\|^2 - \frac{\bar{\sigma}}{(g_0, g_0)}(h - g, g_0) - \frac{\sigma}{(g_0, g_0)}(g_0, h - g) + \frac{|\sigma|^2}{(g_0, g_0)} =$$

$$= \|h - g\|^2 - \frac{|\sigma|^2}{(g_0, g_0)},$$

so that

$$\|h - g^*\| < \|h - g\|,$$

which contradicts (1).

From the proof it follows that h has a representation of the form[6]

$$h = g + f,$$

where $g \in G$ and f is orthogonal to G (in symbols, $f \perp G$). It follows easily that

$$\|h\|^2 = \|g\|^2 + \|f\|^2.$$

It is natural to call g the *component* of h in the subspace G or the *projection* of h on G.[7]

[6] Translator's Note: It is easy to prove that this representation is unique. The author uses the uniqueness later.

[7] This contrasts with the situation in analytic geometry, where a projection is a number and a component is a vector. Here projection and component are equivalent terms.

We denote by F the set of all vectors f orthogonal to the subspace G. We show that F is closed, so that F is a subspace. In fact, let $f_n \in F$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$. Then $(f_n, g) = 0$ and

$$(f, g) = (f - f_n, g).$$

In absolute value, the right member does not exceed

$$\| f - f_n \| \cdot \| g \|,$$

which converges to zero as $n \to \infty$. Hence, $(f, g) = 0$, so that $f \in F$ and the manifold F is closed.

The subspace F is called the *orthogonal complement* of G and is expressed by

(2) $$F = H \ominus G.$$

It is easy to see that

(2′) $$G = H \ominus F.$$

Both relations (2) and (2′) are expressed by the equation

$$H = G \oplus F,$$

because H is the so-called direct sum of the subspaces F and G (in the given case, the *orthogonal sum*).

In general, a set $M \subset H$ is called the *direct sum* of a finite number of linear manifolds $M_k \subset H$ $(k = 1, 2, 3, \ldots, n)$ and is expressed by

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

if each element $g \in M$ is represented uniquely in the form

$$g = g_1 + g_2 + \ldots + g_n$$

where $g_k \in M_k$ $(k = 1, 2, 3 \ldots, n)$. It is evident that M is also a linear manifold.

It will be necessary for us to consider direct sums of an infinite number of linear manifolds only in cases for which the manifolds are pairwise orthogonal subspaces of the given space. This is done as follows.

DEFINITION: *Let* $\{H_\alpha\}$ *be a countable or uncountable class of pairwise orthogonal subspaces of* H. *Their orthogonal sum*

$$\sum_\alpha \oplus H_\alpha$$

is defined as the closed linear envelope of the set of all finite sums of the form

$$H_{\alpha'} \oplus H_{\alpha''} \oplus \ldots .$$

Often it is necessary to determine the projection of a vector on a finite dimensional subspace. We consider this question in some detail. Let G be an n-dimensional subspace and let

(3) $$g_1, g_2, \ldots, g_n$$

be n linearly independent elements of G. Since any $n+1$ elements of G are linearly dependent, each vector $g' \in G$ can be represented (uniquely) in the form

$$g' = \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_n g_n.$$

In other words, G is the linear envelope of the set of vectors (3).

We choose an arbitrary vector $h \in H$ and denote by g its projection on G. The vector g has a unique representation,

$$g = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n.$$

According to the definition of a projection, the difference $h - g = f$ must be orthogonal to the subspace G, i.e., f is orthogonal to each of the vectors $g_1, g_2, \ldots, g_n$. Therefore,

$$(4)\quad (f, g_k) \equiv (h, g_k) - \alpha_1 (g_1, g_k) - \alpha_2 (g_2, g_k) - \ldots - \alpha_n (g_n, g_k) = 0 \qquad (k = 1, 2, 3, \ldots, n).$$

This is a system of n linear equations in the unknowns $\alpha_1, \alpha_2, \ldots, \alpha_n$. We have shown that it has a unique solution for each vector h. Therefore, the determinant of this system is different from zero[8]. This determinant

$$\Gamma(g_1, g_2, \ldots, g_n) = \begin{vmatrix} (g_1, g_1) & (g_2, g_1) & \ldots & (g_n, g_1) \\ (g_1, g_2) & (g_2, g_2) & \ldots & (g_n, g_2) \\ \ldots & \ldots & & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (g_1, g_n) & (g_2, g_n) & \ldots & (g_n, g_n) \end{vmatrix}$$

is called the *Gram determinant* of the vectors $g_1, g_2, \ldots, g_n$. It is easy to see that if the vectors $g_1, g_2, \ldots, g_n$ are linearly dependent, then the Gram determinant is equal to zero. Hence, for the linear independence of the vectors it is necessary and sufficient that their Gram determinant be different from zero.

We proceed to determine the number

$$\delta = \min_{g' \in G} \| h - g' \|.$$

We shall express δ by means of the Gram determinant. As above let g be the projection of h on G and let $f = h - g$. Then $\delta = \| f \| = \| h - g \|$ and

$$\delta^2 = (f, f) = (f, h),$$

[8] Translator's Note: In (4), suppose that $(h, g_k) = 0$ for $k = 1, 2, 3, \ldots, n$. In other words, let $h \perp G$. Then $\| g \|^2 = \|\alpha_1 g_1 + \ldots + \alpha_n g_n\|^2 = \sum_{k=1}^{n} \bar{\alpha}_k [(\alpha_1 g_1, g_k) + \ldots + (\alpha_n g_n, g_k)] = \sum_{k=1}^{n} \bar{\alpha}_k (h, g_k) = 0$ so that $g = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n = 0$. Since $g_1, \ldots, g_n$ are linearly independent, $\alpha_1 = \alpha_2 = \ldots \alpha_n = 0$. This proves that the homogeneous system $\alpha_1 (g_1, g_k) + \ldots + \alpha_n (g_n, g_k) = 0$ $(k = 1, 2, 3, \ldots, n)$ has only the trivial solution $\alpha_1 = \ldots = \alpha_n = 0$. Consequently, the determinant of the system is not zero.

since $(f, g) = 0$. Let $g = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n$, where the g_k are as in equation (3), to obtain

$$\delta^2 = (h, h) - \alpha_1(g_1, h) - \alpha_2(g_2, h) - \ldots - \alpha_n(g_n, h). \tag{5}$$

The determination of δ^2 is reduced to the elimination of the quantities α_i from equations (4) and (5). This elimination yields

$$\begin{vmatrix} (h,h) - \delta^2 & (g_1,h) & (g_2,h) & \ldots & (g_n,h) \\ (h,g_1) & (g_1,g_1) & (g_2,g_1) & \ldots & (g_n,g_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (h,g_n) & (g_1,g_n) & (g_2,g_n) & \ldots & (g_n,g_n) \end{vmatrix} = 0.$$

Hence,

$$\delta^2 = \frac{\Gamma(h, g_1, g_2, \ldots, g_n)}{\Gamma(g_1, g_2, \ldots, g_n)}. \tag{6}$$

This is the formula we wished to obtain.

Since $\Gamma(g_1) = (g_1, g_1) > 0$ (for $g_1 \neq 0$), it follows from formula (6) that the Gram determinant of linearly independent vectors is always positive. This fact can be regarded as a generalization of the Cauchy-Bunyakovski inequality, which asserts that

$$\Gamma(g_1, g_2) > 0$$

for linearly independent vectors g_1 and g_2.

8. Orthogonalization of a Sequence of Vectors

Two sets M and N of vectors in H are said to be *equivalent* if their linear envelopes coincide. Therefore, the sets M and N are equivalent if and only if each element of one of these sets is a linear combination of a finite number of vectors belonging to the other set.

If the elements of the set M are pairwise orthogonal vectors, and if each of the vectors is *normalized*, i.e., if each has norm equal to one, then the set M is called an *orthonormal system*. If, in addition, the set M is countable, then it is also called an *orthonormal sequence*.

Suppose given a finite or infinite sequence of independent vectors

$$g_1, g_2, \ldots, g_n, \ldots . \tag{1}$$

We show how to construct an equivalent orthonormal sequence of vectors

$$e_1, e_2, \ldots, e_n, \ldots . \tag{2}$$

For the first vector, we take

$$e_1 = \frac{g_1}{\|g_1\|},$$

the norm of which is equal to one. The vectors e_1 and g_1 generate the same

(one dimensional) subspace E_1. The vector e_2 is constructed in two steps. First, we subtract from the vector g_2 its projection on E_1 to get

$$h_2 = g_2 - (g_2, e_1)e_1,$$

which is orthogonal to the subspace E_1. Since the vectors (1) are linearly independent, g_2 does not belong to E_1, so that $h_2 \neq 0$. Now let

$$e_2 = \frac{h_2}{\| h_2 \|}.$$

The vectors e_1 and e_2 generate the same (two dimensional) subspace E_2 as do the vectors g_1 and g_2. We now construct the vector e_3. First, we subtract from g_3 its projection on E_2 to get

$$h_3 = g_3 - (g_3, e_1)e_1 - (g_3, e_2)e_2,$$

which is different from zero and orthogonal to the subspace E_2, i.e., h_3 is orthogonal to each of the vectors e_1 and e_2. Next we let

$$e_3 = \frac{h_3}{\| h_3 \|}.$$

We continue in the same way. If the vectors

$$e_1, e_2, \ldots, e_n$$

have been constructed, then we let

$$h_{n+1} = g_{n+1} - \sum_{k=1}^{n} (g_{n+1}, e_k)\, e_k,$$

and

$$e_{n+1} = \frac{h_{n+1}}{\| h_{n+1} \|}.$$

The method described is called *orthogonalization.*[9]

In the solution of many problems concerning manifolds generated by a given sequence of vectors, preliminary orthogonalization of the sequence turns out to be very useful. We illustrate this in the problem considered in the preceding paragraph. That problem concerned the determination of the distance from a point $h \in H$ to a linear manifold G, which was the closed linear envelope of the given sequence (1). We shall show how elegantly this problem is solved if the system (1) is orthogonalized beforehand.

[9] Often, in particular cases, one does not bother to normalize the system (2) of pairwise orthogonal elements which is equivalent to the system (1). The transition from (1) to such an orthogonal system likewise is called orthogonalization. (See Section 11 below.)

Assume given the orthogonal sequence (2) and a vector $h \in H$. For each integer n, the vector h can be expressed in the form

$$h = \sum_{k=1}^{n} (h, e_k)\, e_k + f_n$$

where the vector f_n is orthogonal to each of the vectors $e_1, e_2, \ldots, e_n$. The vector

$$s_n = \sum_{k=1}^{n} (h, e_k)\, e_k \tag{3}$$

belongs to the set of vectors

$$\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n \tag{4}$$

and, of these vectors, s_n is nearest to the vector h. The distance from s_n to h is

$$\delta_n = \min_{\lambda_k} \| h - \lambda_1 e_1 - \lambda_2 e_2 - \ldots - \lambda_n e_n \| = \| f_n \| = \tag{5}$$

$$= \sqrt{\|h\|^2 - \sum_{k=1}^{n} |\, (h, e_k)\, |^2}.$$

This is the distance from the point h to the linear envelope G_n of the set consisting of the first n vectors of the sequence (2). If instead of the linear combination of the nth order (4), we wish to find the linear combination of the $(n+1)$th order,

$$\mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_{n+1} e_{n+1}, \tag{4'}$$

which is nearest to the vector h, then we must take the vector

$$s_{n+1} = \sum_{k=1}^{n+1} (h, e_k)\, e_k.$$

Thus, we do not change the coefficients in the linear combination (3). Rather, we merely add one more term,

$$(h, e_{n+1}) e_{n+1}$$

to the right member of (3).

These considerations show that, being given the infinite orthonormal sequence (2), it is appropriate to associate with each vector $h \in H$ the *infinite series*

$$\sum_{k=1}^{\infty} (h, e_k) e_k. \tag{6}$$

Equation (5) yields the important inequality

$$\sum_{k=1}^{\infty} |\, (h, e_k)\, |^2 \leqq \| h \|^2. \tag{7}$$

The convergence of the series

$$\sum_{k=1}^{\infty} |\, (h, e_k)\, |^2$$

implies that

$$\|s_n - s_m\|^2 = \sum_{k=m+1}^{n} |(h, e_k)|^2 \qquad (m < n)$$

converges to zero as $m, n \to \infty$, i.e., the series (6) converges in H.[10] We see that the square of the distance from the point h to the manifold G is

$$\|h\|^2 - \sum_{k=1}^{\infty} |(h, e_k)|^2$$

and that the vector h belongs to the manifold G if and only if there is equality in formula (7).

We shall say that a system of vectors is *closed* in H if its linear envelope is dense in H. From our considerations, the orthonormal system (2) is closed in H if and only if

$$\|h\|^2 = \sum_{k=1}^{\infty} |(h, e_k)|^2 \tag{8}$$

for each $h \in H$. Following V. A. Steklov,[11] we call this equality the *closure relation.*[12] We show next that if the Parseval relation holds for each vector $h \in H$, then for any pair of vectors $g, h \in H$, the general Parseval relation

$$(g, h) = \sum_{k=1}^{\infty} (g, e_k)(e_k, h) \tag{9}$$

holds. In fact, we have the Parseval relation for each vector $g + \lambda h$:

$$\|g + \lambda h\|^2 = \sum_{k=1}^{\infty} |(g + \lambda h, e_k)|^2,$$

which yields

$$(g, g) + \lambda(h, g) + \bar{\lambda}(g, h) + |\lambda|^2 (h, h) =$$

$$= \sum_{k=1}^{\infty} \{|(g, e_k)|^2 + \lambda(h, e_k)(e_k, g) + \bar{\lambda}(g, e_k)(e_k, h) + |\lambda|^2 |(h, e_k)|^2\}$$

and

$$\lambda(h, g) + \bar{\lambda}(g, h) = \lambda \sum_{k=1}^{\infty} (h, e_k)(e_k, g) + \bar{\lambda} \sum_{k=1}^{\infty} (g, e_k)(e_k, h).$$

Since λ is arbitrary, equation (9) follows.

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[10] We remark that convergence in H of any series $f_1 + f_2 + \ldots$, where $(f_i, f_k) = 0$ $(i \neq k)$ is equivalent to the convergence of the numerical series $\|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2 + \ldots$.

[11] V. A. Steklov showed for the first time the important significance of the closure relation in various problems of analysis and mathematical physics. Before the work of V. A. Steklov, the relation under consideration was known only for systems of trigonometric functions (the so-called Parseval-Liapunov equation).

[12] Translator's Note: Following rather common English practice, we shall refer to (8) henceforth as the *Parseval relation.* Some authors refer to it as the completeness relation.

9. Complete Orthonormal Systems

The vectors of an orthonormal system cannot be linearly dependent. Therefore, in n-dimensional Euclidean space each orthonormal system of vectors contains at most n vectors.

We say that an orthonormal system M is *complete* in H if M is not contained in a larger orthonormal system in H, i.e., if there is no nonzero vector in H which is orthogonal to every vector of the system M. In Euclidean n-space any orthonormal system of n vectors is complete. In Hilbert space a complete orthonormal system contains an infinite number of elements, and there arises the problem of the cardinality of such systems. This problem is solved easily for separable spaces. We begin with them.

THEOREM 1: *If the space* H *is separable, then every orthonormal system of vectors in* H *consists of a finite or countable number of elements.*

Proof: Let

$$f_1, f_2, f_3, \ldots \tag{1}$$

be a sequence of vectors which is dense in H, and let M be an orthonormal system of vectors. We proceed to show that M can be enumerated. Let e and e' be distinct vectors in M. From (1) choose vectors f_k and $f_{k'}$ such that

$$\|e - f_k\| < \tfrac{1}{2}\sqrt{2} \tag{2}$$

and similarly for e' and k'. We show that $k' \neq k$. In fact,

$$\|e - e'\|^2 = \|e\|^2 + \|e'\|^2 = 2$$

so that

$$\sqrt{2} = \|e - e'\| \leq \|e - f_k\| + \|e' - f_k\| < \tfrac{1}{2}\sqrt{2} + \|e' - f_k\|.$$

Therefore,

$$\|e' - f_k\| > \tfrac{1}{2}\sqrt{2}$$

so that $f_{k'} \neq f_k$ and $k \neq k'$. Thus, we can associate with each vector of M a different integer k. This proves that the set M is finite or countable.

The existence of a nondenumerable orthonormal system of vectors in H implies that the space is not separable. An important example of this kind will be considered later.

THEOREM 2: *An infinite orthonormal sequence*

$$e_1, e_2, e_3, \ldots \tag{3}$$

is complete in H *if and only if the sequence is closed in* H.

Proof: (A) Let the system (3) be closed in H. Then, for each vector of H, the Parseval relation holds. Assume that the system (3) is not complete, and denote by h a nonzero vector which is orthogonal to each of the vectors (3). Thus, $(h, e_k) = 0$ $(k = 1, 2, 3, \ldots)$ and the Parseval relation for h reduces to the contradiction

$$0 \neq \|h\|^2 = 0.$$

(B) We suppose now that the system (3) is complete. We choose an arbitrary vector $h \in H$ and consider the sequence of vectors

$$s_n = \sum_{k=1}^{n} (h, e_k) e_k \qquad (n = 1, 2, 3, \ldots).$$

By the preceding section, $\{s_n\}$ is fundamental, which implies that it converges to some vector g. Then

$$(g, e_k) = \lim_{n \to \infty} (s_n, e_k) = (h, e_k) \qquad (k = 1, 2, 3, \ldots) \tag{4}$$

and g belongs to the closed linear envelope of the sequence (3). Consequently, the Parseval relation is valid for g:

$$\| g \|^2 = \sum_{k=1}^{\infty} | (g, e_k) |^2 = \sum_{k=1}^{\infty} | (h, e_k) |^2. \tag{5}$$

It follows from (4) that the vector $g - h$ is orthogonal to each vector of the sequence (3). The assumption that this sequence is complete implies that $g - h = 0$, so that $g = h$, and (5) takes the form

$$\| h \|^2 = \sum_{k=1}^{\infty} | (h, e_k) |^2.$$

We have shown that for an arbitrary vector $h \in H$, the Parseval relation holds. Thus, it is proved that (3) is closed in H.

THEOREM 3: *The space* H *contains a complete orthonormal sequence if and only if it is separable.*

Proof: (A) We assume that the space H is separable, and let N denote a countable set of vectors which is dense in H. Deleting from the sequence N any vector which is a linear combination of the preceding ones, and orthogonalizing the resulting sequence, we obtain an orthonormal sequence M. This sequence is complete. For, let the vector $h \in H$ be orthogonal to each element of the sequence M. Then h is orthogonal to each vector of N. For each $\varepsilon > 0$ there exists a vector $f \in N$ such that

$$\| h - f \| < \varepsilon$$

which implies that

$$\| h \|^2 = (h, h) = (h - f, h) \leqq \| h - f \| \cdot \| h \| < \varepsilon \| h \|$$

and

$$\| h \| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $h = 0$ and the orthonormal sequence M is complete in H.

(B) We assume that (3) is a complete orthonormal sequence in H. Let N be the set of all linear combinations of the form

$$\gamma_1^{(n)} e_1 + \gamma_2^{(n)} e_2 + \ldots + \gamma_n^{(n)} e_n \qquad (n = 1, 2, 3, \ldots),$$

where $\gamma_k^{(r)} = \alpha_k^{(r)} + i\beta_k^{(r)}$, and $\alpha_k^{(r)}$, $\beta_k^{(r)}$ are rational numbers. The set N is countable. For each $h \in H$ and each $\varepsilon > 0$ there exists an integer n such that

$$\left\| h - \sum_{k=1}^{n} (h, e_k)\, e_k \right\| < \frac{\varepsilon}{2}.$$

It is possible to approximate the complex numbers (h, e_k), $(k = 1, 2, 3, \ldots, n)$, by numbers of the form $\gamma_k^{(n)}$ such that

$$\left\| \sum_{k=1}^{n} \{(h, e_k) - \gamma_k^{(n)}\}\, e_k \right\| < \frac{\varepsilon}{2}.$$

Thus, there exists a vector,

$$f = \sum_{k=1}^{n} \gamma_k^{(n)} e_k,$$

in N, for which

$$\| h - f \| < \varepsilon,$$

and this implies that H is separable.

The question of the cardinality of a complete orthonormal system in a separable space now can be answered completely: every complete orthonormal system in a separable space is necessarily an infinite sequence - - - a so-called *orthonormal basis* of the space.

Now we consider arbitrary Hilbert spaces. First, we remark that whatever the cardinality of an orthonormal system M, each vector f has no more than a countable set of nonzero projections on the elements of the system M. This follows from the fact that for any sequence of elements $e', e'', e''', \ldots$ of M the inequality

$$|(f, e')|^2 + |(f, e'')|^2 + |(f, e''')|^2 + \ldots \leqq \|f\|^2$$

holds, which shows that it is possible to enumerate the set of all nonzero numbers (f, e) with $e \in M$. Further, we have

THEOREM 4: *Any two complete orthonormal systems in a Hilbert space have the same cardinal number.*

Proof: Let M and N be two orthonormal systems, each complete in H, with cardinalities $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Choose $e \in M$. At least one of the scalar products (e, f), $f \in N$, is different from zero because otherwise it would be possible to extend the orthonormal system N by appending to it the vector e. On the other hand, by the remark of the previous paragraph, there exists no more than a countable set of elements $f \in N$ for which $(e, f) \neq 0$. We denote these elements by

$$f_1, f_2, \ldots, f_n \qquad (1 \leqq n \leqq \infty). \tag{6}$$

We define a function φ with domain M such that $\varphi(e)$ is the set of vector (6), for which $(e, f) \neq 0$. The function φ is at least single valued and a most countably valued. This function maps M onto a set of countabl subsets of N. Each $f^* \in N$ satisfies $f^* \in \varphi(e^*)$ for some $e^* \in M$, since fc each $f^* \in N$ there exists an element $e^* \in M$ which is not orthogonal to th element f^*. From what has been said it follows that

$$\mathfrak{m} \geqq \mathfrak{n}.$$

Reversing the roles of the systems M and N, we get

$$\mathfrak{n} \geqq \mathfrak{m}.$$

And so

$$\mathfrak{m} = \mathfrak{n},$$

and the theorem is proved.[13] The following definition is based on thi theorem.

DEFINITION: *The dimension of a Hilbert space* H *is the cardinality of complete orthonormal system in* H.

It is not necessary to make a separate definition of the dimensio of a subspace $G \subset H$. The dimension of an arbitrary linear manifol $L \subset H$ is defined as the dimension of the corresponding subspace $\bar{L}$.

If two Hilbert spaces H and H′ have the same dimension, then the are *isomorphic* in the sense that there exists a one-to-one correspondenc between H and H′ having the following property. If the elements $f, g \in H$ correspond to the elements $f', g' \in H'$, respectively, then (1) $\alpha f + \beta$ corresponds to $\alpha f' + \beta g'$ and (2) $(f, g)_H = (f', g')_{H'}$. In fact, since th spaces H and H′ have the same dimension they possess complete ortho normal systems of identical cardinalities. We choose any one-to-on correspondence between the elements of these two orthonormal system and extend this correspondence to the linear envelopes of the orthogona systems under consideration in such a way that condition (1) is satisfied Then condition (2) is automatically satisfied, which permits one to get, b passage to the limit, the required correspondence for all the elements o the spaces H and H′.

From the proof it follows that any separable space is isomorphic t the space l^2. It is evident that two Hilbert spaces of different dimension are not isomorphic. Therefore, two abstract Hilbert spaces (similarly, tw abstract Euclidean spaces) differ from each other only in their dimension

[13] The actual construction of a complete orthonormal system in a non-separable space r quires transfinite induction.

10. The Space L^2

Let (a, b) denote a finite or infinite interval[14] on the real axis. We denote by $L^2(a, b)$ (or simply by L^2) the set of all complex valued Lebesgue measurable functions f defined on (a, b) such that $|f|^2$ is Lebesgue integrable on (a, b). We do not regard as distinct elements of L^2 a pair of functions which differ only on a set of measure zero.

It follows by means of the inequality

$$|\alpha + \beta|^2 \leqq 2|\alpha|^2 + 2|\beta|^2$$

that $f + g \in L^2$ whenever $f, g \in L^2$. Furthermore, for each complex number λ and each $f \in L^2$, it follows that $\lambda f \in L^2$. Thus, L^2 is a linear space and the zero element is a function which is equal to zero almost everywhere in (a, b). In this linear space the scalar product is defined by the formula

$$(f, g) = \int_a^b f(t)\overline{g(t)}\,dt.$$

The existence of the integral on the right side is a consequence of the inequality

$$|\alpha\beta| \leqq \tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2.$$

In the present case, the inequality

$$|(f, g)| \leqq \|f\| \cdot \|g\|$$

has the form

$$\left|\int_a^b f(t)\overline{g(t)}\,dt\right| \leqq \sqrt{\int_a^b |f(t)|^2\,dt}\,\sqrt{\int_a^b |g(t)|^2\,dt}.$$

This inequality was obtained by Bunyakovski for Riemann integrals.

Now we show that L^2 is complete, from which it will follow that L^2 is a Hilbert space. Let the sequence of functions $f_n \in L^2$ $(n = 1, 2, 3, \ldots)$ be fundamental, i.e., let

$$\lim_{m, n \to \infty} \int_a^b |f_n(t) - f_m(t)|^2\,dt = 0.$$

Then there exists an infinite sequence of integers

$$k_1 < k_2 < k_3 < \ldots < k_r < \ldots,$$

[14] Instead of an interval, the domain of definition of the function could be any measurable set (of finite or infinite measure) on the real axis, on the plane, or in Euclidean n-space.

for which

$$\int_a^b |f_{k_{r+1}}(t) - f_{k_r}(t)|^2 dt < \frac{1}{8^r} \qquad (r = 1, 2, 3, \ldots).$$

From this inequality it follows that the set of points of the interval (a, b) for which

$$|f_{k_{r+1}}(t) - f_{k_r}(t)| \geqq \frac{1}{2^r}$$

has measure less than $\frac{1}{2^r}$. For $s = 1, 2, 3, \ldots$, let I_s denote the set of points $t \in (a, b)$ such that

$$|f_{k_{s+1}}(t) - f_{k_s}(t)| < \frac{1}{2^s},$$

$$|f_{k_{s+2}}(t) - f_{k_{s+1}}(t)| < \frac{1}{2^{s+1}},$$

$$\ldots\ldots\ldots\ldots\ldots$$

The complement of I_s with respect to the interval $I = (a, b)$ has measure

$$m(I - I_s) < \sum_{n=s}^{\infty} \frac{1}{2^n} = \frac{1}{2^{s-1}}.$$

Since $I_n \subset I_{n+1} \subset \ldots$, $\lim_{n\to\infty} I_n = I^*$ exists and

$$m(I - I^*) = 0.$$

The sequence $\{f_{k_r}(t)\}_{r=1}^{\infty}$ converges uniformly on each set I_s. This follows from the inequality

$$|f_{k_n}(t) - f_{k_m}(t)| \leqq \sum_{r=m}^{n-1} |f_{k_{r+1}}(t) - f_{k_r}(t)| < \sum_{r=m}^{n-1} \frac{1}{2^r} < \frac{1}{2^{m-1}} \quad (n > m > s)$$

which is valid for $t \in I_s$. Consequently, $\{f_{k_r}(t)\}_{r=1}^{\infty}$ converges on I^* (i.e., almost everywhere in I). Let

$$f(t) = \begin{cases} \lim_{r\to\infty} f_{k_r}(t) & (t \in I^*), \\ 0 & (t \in I - I^*). \end{cases}$$

Since $\{f_n\}$ was assumed to be a fundamental sequence, for each $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$\int_{I_s(a)} |f_m(t) - f_{k_r}(t)|^2 dt \leqq \|f_m - f_{k_r}\|^2 < \varepsilon$$

for $m, k_r > N(\varepsilon)$, where $I_s(a) = I_s$ if the interval (a, b) is finite and $I_s(a) = I_s \cap (-a, a)$ otherwise. In $I_s(a)$, convergence of the sequence $\{f_{k_r}(t)\}_{r=1}^{\infty}$ is uniform. Therefore, in the integral, passage to the limit is legitimate and we obtain

$$\int_{I_s(a)} |f_m(t) - f(t)|^2 dt \leqq \varepsilon \qquad (m > N(\varepsilon)).$$

It follows that

$$\int_{I_s} |f_m(t) - f(t)|^2 dt \leqq \varepsilon,$$

where s is arbitrary. Hence

$$\int_a^b |f_m(t) - f(t)|^2 dt \leqq \varepsilon.$$

This implies that $f_m - f \in L^2$, so that $f \in L^2$. Since $\varepsilon > 0$ is arbitrary, we have proved also that

$$\lim_{m \to \infty} \int_a^b |f_m(t) - f(t)|^2 dt = 0 .$$

In the process of the proof, we have obtained the following fact: if the sequence $f_n \in L^2$ $(n = 1, 2, 3, \ldots)$ converges to f in norm, i.e., if $\|f_n - f\| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{f_{k_r}(t)\}_{r=1}^{\infty}$ which converges to $f(t)$ almost everywhere. Furthermore, if a proper set of arbitrarily small measure is removed from the interval (a, b), then in the remaining set the subsequence $\{f_{k_r}(t)\}_{r=1}^{\infty}$ converges to $f(t)$ uniformly.

We remark that it is possible to consider the space $L^2(a, b)$ as a subspace of $L^2(a_1, b_1)$ if $a_1 \leqq a < b \leqq b_1$ and, in particular, as a subspace of $L^2(-\infty, \infty)$. For this, it is necessary to extend each function $f \in L^2(a, b)$ beyond the limits of the interval (a, b) by defining $f(t)$ to be zero for t outside of (a, b).

Convergence in the metric space L^2 is called *convergence in the mean* and is denoted by

$$f(t) = \underset{n \to \infty}{\text{l. i. m.}} f_n(t),$$

if

$$\lim_{n \to \infty} \int_a^b |f(t) - f_n(t)|^2 dt = 0$$

(l. i. m.) is an abbreviation for limes in medio, i.e. *limit in the mean*).

11. Complete Orthonormal Systems in L^2

In the present paragraph, we show that there exist complete orthonormal sequences in $L^2(a, b)$, where a and b are finite or infinite. Hence, by Theorem 2, Section 9, it will follow that the space L^2 is separable. It would be possible to prove this latter fact immediately. In fact, using the definition of the Lebesgue integral, it is not difficult to prove that the linear envelope of the set of functions f such that $f \equiv 1$ on some finite interval and $f \equiv 0$ outside, is complete in L^2. Hence, the separability of L^2 follows.

A. We begin with the space $L^2(0, 2\pi)$. In this space, the functions

$$\frac{1}{\sqrt{2\pi}} e^{ikt} \qquad (\pm k = 0, 1, 2, \ldots)$$

form an orthonormal system. We wish to show that this *trigonometric system* is complete. Assume there exists $f \in L^2(0, 2\pi)$ such that

$$\int_0^{2\pi} |f(t)|\, dt \neq 0$$

and

$$(1) \qquad \int_0^{2\pi} f(t) e^{-ikt} dt = 0 \qquad (\pm k = 1, 2, 3, \ldots).$$

It follows by means of integration by parts that the function

$$F(t) = \int_0^t f(t)\, dt$$

satisfies the equations

$$(2) \qquad \int_0^{2\pi} \{F(t) - C\} e^{-ikt} dt = 0 \qquad (\pm k = 1, 2, 3, \ldots)$$

for any constant C. We specify this constant so that equation (2) holds also for $k = 0$. Since the function

$$\Phi(t) = F(t) - C$$

is continuous, the well-known theorem of Weierstrass applies: for each $\varepsilon > 0$ there exists a trigonometric sum

$$\sigma(t) = \sum_{k=-n}^{n} A_k e^{ikt}$$

such that

$$|\Phi(t) - \sigma(t)| < \varepsilon.$$

Therefore, using relation (2), we obtain

$$\int_0^{2\pi} |\Phi(t)|^2 dt = \int_0^{2\pi} \overline{\Phi(t)} \{\Phi(t) - \sigma(t)\} dt \leqq$$

$$\leqq \varepsilon \int_0^{2\pi} |\Phi(t)| dt \leqq \varepsilon \sqrt{2\pi} \sqrt{\int_0^{2\pi} |\Phi(t)|^2 dt}$$

whence

$$\int_0^{2\pi} |\Phi(t)|^2 dt \leqq 2\pi\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, this implies that $\Phi(t) = 0$, so that $F(t) = C$ and $f(t) = 0$ almost everywhere. Thus, the completeness of the trigonometric system is proved.

B. We consider now the space $L^2(a, b)$, where (a, b) is an arbitrary finite interval. The orthogonalization of the sequence of functions

$$1, t, t^2, \ldots$$

yields the sequence of polynomials

$$C_k \frac{d^k \{(t-a)(t-b)\}^k}{dt^k} \qquad (k = 0, 1, 2, \ldots),$$

where C_k are certain positive constants. These are the well-known *Legendre polynomials*. They are usually considered for $a = -1$, $b = 1$. The completeness of this orthonormal system may be proved in the same way as the completeness of the trigonometric system.

C. We consider the space $L^2(-\infty, \infty)$. The orthogonalization of the system

$$e^{-\frac{t^2}{2}}, te^{-\frac{t^2}{2}}, t^2 e^{-\frac{t^2}{2}}, \ldots$$

yields the sequence of *Tchebysheff-Hermite functions*,

$$\varphi_k(t) = (-1)^k e^{\frac{t^2}{2}} \frac{d^k e^{-t^2}}{dt^k} = H_k(t) e^{-\frac{t^2}{2}} \qquad (k = 0, 1, 2, \ldots),$$

where $H_k(t)$ is the so-called *Tchebysheff-Hermite polynomial* of degree k. The Tchebysheff-Hermite functions satisfy the relations

$$\int_{-\infty}^{\infty} \varphi_k(t) \varphi_m(t) dt = \begin{cases} 0 & (k \neq m), \\ 2^m m! \sqrt{\pi} & (k = m), \end{cases}$$

so that they are pairwise orthogonal but not normalized. We prove next that the sequence of Tchebysheff-Hermite functions is complete. Assume there exists a nonzero function $f \in L^2(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} f(t)\,\varphi_k(t)\,dt = 0 \qquad (k = 0, 1, 2, \ldots)$$

or, equivalently, such that

$$\int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} t^k\,dt = 0 \qquad (k = 0, 1, 2, \ldots). \tag{3}$$

We introduce the function

$$F(z) = \int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} e^{itz}\,dt$$

which, it is evident, exists for every complex z. The function $F(z)$ has the finite derivative

$$F'(z) = \int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} e^{itz}\,it\,dt.$$

Since this equation holds everywhere in the complex plane, $F(z)$ is an entire function. But, by (3)

$$F^{(k)}(0) \equiv \int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} (it)^k\,dt = 0 \qquad (k = 0, 1, 2, \ldots)$$

so that $F(z)$ is identically zero. Therefore,

$$\int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} e^{itx}\,dt = 0 \qquad (-\infty < x < \infty).$$

Multiplying this equality by e^{-ixy}, where y is real, and integrating with respect to x from $-\omega$ to ω, we get

$$\int_{-\infty}^{\infty} f(t)\,e^{-\frac{t^2}{2}} \frac{\sin \omega (t-y)}{t-y}\,dt = 0,$$

which is valid for every real y and ω. Hence, as is proved in analysis courses, it follows that $f(t) = 0$ almost everywhere, and this contradicts our original assumption.

D. In the space $L^2(0, \infty)$, we have the orthonormal system of *Tchebysheff-Laguerre functions*

$$\psi_k(t) = \frac{e^{-\frac{t}{2}} L_k(t)}{k!} \qquad (k = 0, 1, 2, \ldots),$$

where the $L_k(t)$ are the *Tchebysheff-Laguerre polynomials* which are defined by the formulas

$$L_k(t) = e^t \frac{d^k}{dt^k}(t^k e^{-t}) \qquad (k = 0, 1, 2, \ldots).$$

The completeness of this system can be proved by an argument based on the completeness of the Tchebysheff-Hermite system. We leave this to the reader.

12. The Space L_σ^2

Let a non-decreasing function of bounded variation $\sigma(t)\,(-\infty < t < \infty)$ be given. We assume that it is left-continuous:

$$\sigma(t-0) = \sigma(t).$$

Such a function is often called a distribution function. With the aid of the function $\sigma(t)$ it is possible to construct a measure analogous to the Lebesgue measure but differing from it in that the length $b-a$ of the interval $[a, b]$, $(a \leqq b)$[15] is replaced by the *σ-length* $\sigma(b+0)-\sigma(a)$. Some points may have σ-length different from zero (points of jumps of $\sigma(t)$) and some proper intervals may have σ-length equal to zero (intervals of constancy of $\sigma(t)$). The measure determined by the σ-length is called the *σ-measure*; the *σ-measurable functions* and the corresponding Lebesgue-Stieltjes integral are constructed from it.

We consider the linear space of all σ-measurable functions f for which the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{\infty} |f(t)|^2 \, d\sigma(t)$$

exists, and metrize it by means of the metric generated by the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, d\sigma(t).$$

This linear space is complete, so that it is a Hilbert space. It is denoted by L_σ^2. Special significance is possessed by *characteristic functions*. A characteristic function is equal to one in a certain finite interval of positive σ-length and is equal to zero outside of that interval. We do not exclude improper intervals here. The linear envelope of the set of all characteristic functions is dense in L_σ^2. Using this fact, it is easy to prove that L_σ^2 is separable.

The case of a distribution function $\sigma(t)$ for which each integral

$$s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t) \qquad (k = 0, 1, 2, \ldots)$$

[15] For $a = b$, we get an *improper* interval, i.e., a point.

exists is of great interest. Because of mechanical considerations these integrals are called *moments* of the distribution function $\sigma(t)$. If $\sigma(t)$ has only a finite number of points of increase, then the Stieltjes integral becomes a finite sum. We do not consider this uninteresting case.

Orthogonalizing the sequence,

$$1, t, t^2, \ldots,$$

in L^2_σ, we get a sequence of polynomials $\{P_k(t)\}_{k=1}^{\infty}$ (where $P_k(t)$ is a polynomial of degree k), which satisfy the relations

$$\int_{\infty}^{\infty} P_k(t) P_m(t)\, d\sigma(t) = \begin{cases} 0 & (k \neq m), \\ 1 & (k = m). \end{cases}$$

This polynomial sequence is said to be orthonormal with respect to the distribution function $\sigma(t)$. If the function $\sigma(t)$ is absolutely continuous and if

$$\sigma'(t) = w(t)$$

then the orthonormality relations can be written in the form

$$\int_{-\infty}^{\infty} P_k(t) P_m(t) w(t)\, dt = \begin{cases} 0 & (k \neq m), \\ 1 & (k = m). \end{cases}$$

In this case, it is said that the polynomials $P_k(t)$ are orthonormal with respect to the *weight* $w(t)$. For instance, the Tchebysheff-Hermite polynomials are orthogonal (but not normalized) with respect to the weight e^{-t^2} $(-\infty < t < \infty)$.

If the interval of orthogonality is finite (i.e., if $\sigma(t)$ is constant for $t < a$ and for $t > b$), then the orthogonal polynomials $P_k(t)$ form a complete system. This is proved with the aid of the theorem of Weierstrass in exactly the same way as in the proof of the completeness in $L^2(a, b)$ of the sequence of Legendre polynomials.

If the interval of orthogonality is infinite, then the system of orthogonal polynomials $P_k(t)$ may fail to be complete. We restrict ourselves to one example, which is due to Hamburger.[16] Consider the interval $(0, \infty)$. The orthogonal polynomials on this interval with respect to the weight

$$w(t) = e^{-\frac{\pi\sqrt{t}}{\ln^2 t + \pi^2}}$$

is an incomplete system because the function

$$g(t) = e^{\frac{\ln t}{\ln^2 t + \pi^2}} \sin \frac{\sqrt{t} \ln t + \pi}{\ln^2 t + \pi^2}$$

[16] H. Hamburger [1].

satisfies the relations

$$\int_0^\infty t^k g(t) w(t)\, dt = 0 \qquad (k = 0, 1, 2, \ldots).$$

13. The Space of Almost Periodic Functions

We consider the set of all functions of the form $e^{i\lambda t}$ $(-\infty < t < \infty)$, where the parameter λ is real. We denote by L the linear envelope of this set, i.e., the collection of all "polynomials" of the form

$$\sum_k A_k e^{i\lambda_k t}.$$

Adding to L the limits of sequences of functions of L which are uniformly convergent on the entire real axis, we get a certain set B of continuous functions. As H. Bohr proved, a continuous function $f(t)$ defined on the real axis belongs to the collection B if and only if it is *almost periodic*, i.e., if for each $\varepsilon > 0$ there exists a real number $l = l(\varepsilon)$ such that in every interval of length l there is at least one number τ for which

$$|f(t+\tau) - f(t)| < \varepsilon \qquad (-\infty < t < \infty).$$

We can metrize the linear system L, by defining the scalar product of two polynomials

$$f(t) = \sum_{r=1}^{m} A_r e^{i\lambda_r t},$$

$$g(t) = \sum_{s=1}^{n} B_s e^{i\mu_s t}$$

as

$$(f, g) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\overline{g(t)}\, dt =$$

$$= \lim_{T\to\infty} \sum_{r,s=1}^{m,n} A_r \bar{B}_s \frac{1}{2T} \int_{-T}^{T} e^{it(\lambda_r - \mu_s)}\, dt = \sum_{r,s=1}^{m,n} \delta(\lambda_r, \mu_s) A_r \bar{B}_s$$

where

$$\delta(\lambda, \mu) = \begin{cases} 0 & (\lambda \neq \mu), \\ 1 & (\lambda = \mu). \end{cases}$$

When L is closed by means of the metric generated by this scalar product, we get a certain complete Hilbert space B^2 which contains B as a linear manifold. The space B^2 is not separable. This follows from the fact that in B^2 there exists a continuum of orthonormal vectors $e^{i\lambda t}$ $(-\infty < \lambda < \infty)$, whereas (see paragraph 9) every orthonormal system in a separable space contains a finite or countable number of vectors.

Chapter II

LINEAR FUNCTIONALS AND BOUNDED LINEAR OPERATORS

14. Point Functions

Two kinds of point functions are considered in elementary treatments of 3- and n-dimensional spaces: *scalar functions*, the values of which are (real or complex) numbers, and *vector functions*, which relate the points of a space to other points of the same or another space. In the present book we shall study point functions in Hilbert space. In correspondence with the indicated division of the functions of elementary analysis into scalar functions and vector functions, we introduce in H so-called *functionals* and *operators*. The appropriate definitions follow.

Let D denote a subset of the space H. A function Φ which relates to each point $f \in \mathrm{D}$ a definite complex number $\Phi(f)$ is called a *functional* in the space H with *domain* D. A function T which relates to each element $f \in \mathrm{D}$ a particular element $Tf = g \in \mathrm{H}$ is called an operator[1] in the space H with *domain* D. The set Δ, consisting of all $g = Tf$, where f runs through D, is called the *range* of T. Sometimes we shall denote the domain of a functional Φ by D_Φ and, correspondingly, the domain and range of an operator T by D_T and Δ_T respectively.

The identity operator, i.e., the operator which maps each vector into itself, we shall denote by E. The operator which maps every vector into zero, we shall denote by O.

If the operator T maps each pair of different elements of D into a pair of different elements of Δ, then T has an inverse operator, which maps the elements of Δ into elements of D. The inverse operator is denoted by the symbol T^{-1}, and $T^{-1}g = f$ if and only if $f = Tg$. Moreover,

$$\mathrm{D}_{T^{-1}} = \Delta_T, \qquad \Delta_{T^{-1}} = \mathrm{D}_T.$$

We shall consider two functionals (or operators) to be equivalent if their domains coincide and if for each element of their common domain, the values of these functionals (or operators) coincide.

[1] Sometimes it is necessary to consider also functions which map elements of the space H into elements of some other Hilbert space. These functions are also called operators. They will not be encountered often in this book.

If the domain D_T of the operator T contains the domain D_S of the operator S, i.e., if $D_S \subset D_T$, and if

$$Tf = Sf$$

for each $f \in D_S$, then T is called an *extension* of S and we write

$$S \subset T.$$

The concept of an extension of a functional is defined analogously.

Motivated by the notion of a continuous function, we make the following definition of the *continuity of an operator:* T is continuous at a point $f_0 \in D_T$ if

$$\lim_{f \to f_0} Tf = Tf_0 \qquad (f \in D_T).$$

An equivalent condition is that for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if f satisfies the inequality,

$$\|f - f_0\| < \delta, \quad f \in D_T,$$

then

$$\|Tf - Tf_0\| < \varepsilon.$$

The *continuity of a functional* is defined analogously.

If the element f_0 does not belong to D_T but $\lim Tf = g_0$ exists as $f \to f_0$ with $f \in D_T$, then the operator T can be defined for f_0, by letting $Tf_0 = g_0$. Proceeding in exactly the same way with all such elements f_0, we arrive at the so-called *extension by continuity of the operator* T. This extension is uniquely defined for each operator T. The *extension by continuity of a functional* is defined analogously.

We shall introduce below notation relating to so-called linear functionals and linear operators, which are the basic objects of our study.

Here we consider only the operator analogue of a "function of a function". Let S and T be two operators such that the range of T intersects the domain of S (i.e., let $\Delta_T \cap D_S \neq 0$). In this case we define the *product* ST of the operators S and T as the operator such that

$$STf = S(Tf)$$

for each element f in its domain, which is defined as the set of all $f \in D_T$ for which $Tf \in D_S$. The product TS is defined analogously whenever the set $\Delta_S \cap D_T$ is non-empty. It is clear that ST and TS are not generally equivalent because their domains may be different, and, moreover, even if g is an element belonging to both domains, it is possible that

$$STg \neq TSg.$$

Since it is difficult to give a reasonable completely general definition of the commutativity of two operators, we restrict ourselves here to the

case in which at least one of the two operators S and T is defined everywhere in H. Let $D_S = H$. Then S and T are commutative if

$$ST \subset TS$$

i e , if $f \in D_T$ implies both $Sf \in D_T$ and

$$STf = TSf.$$

In particular, if both operators are defined everywhere in H, then S and T are commutative if and only if $ST = TS$.

15. Linear Functionals

A functional Φ is said to be *linear* if:

(a) its domain D is a linear manifold and

$$\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$$

for $f, g \in D$ and any complex numbers α and β;

(b) the inequality

$$\sup_{f \in D,\ \|f\| \leq 1} |\Phi(f)| < \infty$$

is satisfied.

The left member of this inequality is called the *norm* of the functional Φ and is denoted by the symbol $\|\Phi\|_D$ or, if $D = H$, simply by $\|\Phi\|$.

If $f \in D$ and $f \neq 0$ then, by the definition of the norm of a functional,

$$\left| \Phi\left(\frac{f}{\|f\|}\right) \right| \leq \|\Phi\|_D.$$

Hence, for $f \in D$,

$$|\Phi(f)| \leq \|\Phi\|_D \cdot \|f\|. \tag{1}$$

Relation (1) shows that the linear functional Φ is continuous. In fact, by (1),

$$|\Phi(f) - \Phi(f_0)| = |\Phi(f - f_0)| \leq \|\Phi\|_D \cdot \|f - f_0\|$$

for $f, f_0 \in D$.

From (1) it also follows that, if $f \in D$ and $\|f\| \leq 1$, then

$$|\Phi(f)| \leq \|\Phi\|_D,$$

with strict inequality if $\|f\| < 1$. Therefore, the norm $\|\Phi\|_D$ can be defined by

$$\sup_{f \in D,\ \|f\| = 1} |\Phi(f)| = \|\Phi\|_D, \tag{2}$$

or, equivalently, by[2]

$$\sup_{f \in D} \frac{|\Phi(f)|}{\|f\|} = \|\Phi\|_D. \tag{2'}$$

[2] Translator's Note: The condition $f \neq 0$ should be included in (2'). Similar conditions should be added in many other relations which occur below. This task is left to the reader.

If Φ and Ψ are two linear functionals with domains of definition D_Φ and D_Ψ, then $\alpha\,\Phi + \beta\,\Psi$, where α and β are constants, is also a linear functional, with the intersection $D_\Phi \cap D_\Psi$ of the domains D_Φ and D_Ψ as domain (of course, only the case when $D_\Phi \cap D_\Psi$ contains elements different from $f = 0$ offers much interest).

If the functional Φ satisfies the condition (a) listed above, which states that Φ is *homogeneous and additive*, and if Φ is continuous at any one point $f_0 \in D$, then Φ also satisfies condition (b) above, i.e., Φ is bounded and, therefore, Φ is a linear functional. In fact, if Φ is continuous at f_0 then for each $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|\Phi(h) - \Phi(f_0)| < \delta$$

for $\|h - f_0\| \leqq \varepsilon$ and $h \in D$. For each $f \in D$ such that $f \neq 0$,

$$\Phi(f) = \frac{\|f\|}{\varepsilon}\,\Phi\left(\frac{\varepsilon f}{\|f\|}\right) = \frac{\|f\|}{\varepsilon}\left\{\Phi\left(\frac{\varepsilon f}{\|f\|} + f_0\right) - \Phi(f_0)\right\}.$$

Since the vector $\dfrac{\varepsilon f}{\|f\|} + f_0 = h$ satisfies the relation $\|h - f_0\| = \varepsilon$ we have, for $f \in D$,

$$|\Phi(f)| < \frac{\delta}{\varepsilon}\|f\|;$$

in other words, for $f \in D$ and $f \neq 0$,

$$\frac{|\Phi(f)|}{\|f\|} < \frac{\delta}{\varepsilon}.$$

This proves that Φ is bounded.

If the linear manifold D on which the linear functional Φ is defined is not closed, then it is possible to extend Φ by continuity to the closure of D. This extension, as is easy to see, leads to a unique linear functional with the same norm as the initial functional.

16. The Theorem of F. Riesz

The following theorem of F. Riesz provides a representation for each linear functional in H.

THEOREM: *Each linear functional Φ in the Hilbert space* H *can be expressed in the form*

$$\Phi(h) = (h, f),$$

where f is an element of H *which is uniquely determined by the functional Φ; furthermore,*

$$\|\Phi\| = \|f\|.$$

Proof: We denote by G the set of all elements $g \in H$ for which

$$\Phi(g) = 0.$$

By the linearity of the functional Φ, the set G is a linear manifold. Furthermore, G is closed, so that G is a subspace. In fact, if $g_n \in G$, $n \geqq 1$, and $g_n \to g$ then, by the continuity of Φ,

$$\Phi(g) = \lim_{n\to\infty} \Phi(g_n),$$

so that $\Phi(g)=0$ and $g \in G$. If $G=H$, then the functional Φ is equal to zero everywhere, and the theorem of Riesz is proved by taking $f=0$. We now suppose that $G \neq H$. Then there exists a nonzero element $f_0 \in H \ominus G$. We consider elements of the form

$$\Phi(h)f_0 - \Phi(f_0)\, h,$$

where h runs through H. These elements belong to G because

$$\Phi[\Phi(h)f_0 - \Phi(f_0)\, h] = \Phi(h)\,\Phi(f_0) - \Phi(f_0)\,\Phi(h) = 0.$$

Since $f_0 \in H \ominus G$,

$$(\Phi(h)f_0 - \Phi(f_0)\, h, f_0) = 0$$

and

$$\Phi(h) = \left(h, \frac{\overline{\Phi(f_0)}}{(f_0,f_0)} f_0\right).$$

If we set

$$f = \frac{\overline{\Phi(f_0)}}{(f_0,f_0)} f_0,$$

then it follows from the equality just obtained that

$$\Phi(h) = (h,f).$$

This is the required representation of the functional Φ.

We now prove that f is unique. Assuming the contrary, we have the equation

$$(h,f') = (h,f''),$$

for $h \in H$, where $f' \neq f''$. But this is impossible since the substitution of $h = f' - f''$ yields the contradiction,

$$\|f' - f''\|^2 = 0.$$

It remains to be proved that

$$\|\Phi\| = \|f\|.$$

It follows from the equation

$$\Phi(h) = (h,f)$$

that

$$\|\Phi(h)\| \leqq \|f\| \cdot \|h\|$$

which yields

$$\|\Phi\| \leqq \|f\|.$$

On the other hand, taking $h = f$, we get

$$\Phi(f) = \|f\|^2,$$

whence it follows that

$$\| \Phi \| \geqq \| f \|.$$

Thus the theorem of F. Riesz is proved.

We consider now a linear functional Ψ with domain D_Ψ closed in H. Then D_Ψ is a subspace of H and the theorem of F. Riesz asserts the existence of a unique element $g \in D_\Psi$ such that

$$\Psi(h) = (h, g) \qquad (h \in D_\Psi) \tag{3}$$

and

$$\| \Psi \|_{D_\Psi} = \| g \|.$$

By means of (3), the linear functional Ψ may be extended to the whole space H without increasing the norm.[3] Any other extension of the linear functional Ψ to the whole space H increases the norm of the functional. In fact, if Φ is any extension of Ψ to the whole space, then

$$\Phi(h) = (h, f)$$

and

$$\| \Phi \| = \| f \|.$$

For $h \in D_\Psi$,

$$(h, g) = (h, f)$$

so that $f - g \perp D_\Psi$. Because $g \in D_\Psi$,

$$\| f \|^2 = \| g \|^2 + \| f - g \|^2$$

which implies that

$$\| \Phi \| \geqq \| \Psi \|_{D_\Psi},$$

where there is strict inequality if $f \neq g$.

17. A Criterion for the Closure in H of a Given System of Vectors

According to the definition in Section 8, a system M of vectors is closed in H if it is possible to approximate each $h \in H$ to any degree of accuracy by means of a linear combination of vectors belonging to M.

THEOREM: *In order that the system* M *be closed in* H, *it is necessary and sufficient that a linear functional* Φ *in* H *which vanishes for all* $g \in M$, *be identically equal to zero.*

Proof: The necessity is an immediate consequence of the continuity of the linear functional. In order to prove the sufficiency, let us assume that the system is not closed. Then there exists $\delta > 0$ and a vector $h_0 \in H$ for which

$$\inf_{n,\, a_i} \| h_0 - a_1 g_1 - a_2 g_2 - \ldots - a_n g_n \| = \delta > 0 \qquad (g_i \in M).$$

[3] Since any linear functional can be extended to the whole space without increasing the norm, one usually considers a linear functional as being defined on the whole space when the domain is not specified.

We denote by G the closed linear envelope of the system M. On the basis of Section 6, there exists $g \in G$ such that

$$\| h_0 - g \| = \delta.$$

Let

$$f = h_0 - g.$$

Then $f \perp G$. Consider the functional Φ defined by

$$\Phi(h) = (h, f),$$

the norm of which is equal to $\| f \| = \delta > 0$. This nonzero functional vanishes for each vector of G and, in particular, for each vector of M. Thus, the sufficiency is also proved.

18. A Lemma Concerning Convex Functionals[4]

Definition: *A real functional $p(h)$ in* H *is said to be convex if*

$$p(f + g) \leqq p(f) + p(g) \tag{1}$$

and

$$p(\alpha f) = |\alpha| p(f), \tag{2}$$

for $f, g \in H$ and any complex number α.

From this definition it follows that $p(0) = 0$ and $p(h) \geqq 0$.

LEMMA: *If a convex functional $p(h)$ is lower semicontinuous*, i.e., *if for each $h_0 \in H$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$p(h) - p(h_0) > -\varepsilon$$

for $\| h - h_0 \| < \delta$, then the convex functional is bounded, i.e., *there exists $M > 0$ such that*

$$p(h) \leqq M \| h \|$$

for $h \in H$.

Proof:[5] First, we prove that if the functional is not bounded in the unit sphere ($\| h \| < 1$), then it will not be bounded in the sphere $S(\rho, g)$ with center $g \in H$ and radius $\rho > 0$, where g and ρ are arbitrary. For, assuming that $p(h) < C$ for $\| h - g \| < \rho$, we find that

$$p(h - g) \leqq p(h) + p(-g) = p(h) + p(g) < 2C$$

for $\| h - g \| < \rho$. Consequently, if

$$f = \frac{h - g}{\rho},$$

[4] In this section we follow I. M. Gelfand [1].

[5] This proof does not require that H be a Hilbert space; it goes through if H is any Banach space.

then

$$p(f) = \frac{2C}{\rho}$$

for $f \in S(1, 0)$, so that the functional $p(h)$ is bounded in the unit sphere. In view of property (2) it is sufficient to prove that the functional $p(h)$ is bounded in the sphere $S(1, 0)$. We assume the contrary. Then $p(h)$ is unbounded in every sphere. We choose a point $f_1 \in S(1, 0)$ such that $p(f_1) > 1$. The lower semicontinuity of the functional $p(h)$ implies that there is a sphere $S(\rho_1, f_1) \subset S(1, 0)$ with radius $\rho_1 < \frac{1}{2}$ at all points of which $p(h) > 1$. Since $p(h)$ is unbounded in every sphere, there exists a point $f_2 \in S(\rho_1, f_1)$ and also a sphere $S(\rho_2, f_2) \subset S(\rho_1, f_1)$ with radius $\rho_2 < \frac{1}{2}\rho_1$ in which $p(h) > 2$. Continuing this process, we get an infinite sequence of spheres,

$$S(1, 0) \supset S(\rho_1, f_1) \supset S(\rho_2, f_2) \supset \ldots$$

for which $\rho_n < \frac{1}{2}\rho_{n-1}$, $(n = 1, 2, 3, \ldots; \rho_0 = 1)$, and also $p(h) > n$ if $h \in S(\rho_n, f_n)$. But the sequence of centers $\{f_n\}_1^\infty$ is fundamental and, therefore, converges to some element f. Then $p(f) > n$ for each n, which is impossible. Thus, the lemma is proved.

We remark that this lemma can also be formulated as follows: if a convex functional is lower semicontinuous, then it is continuous.

COROLLARY: *Let $p_k(h)$, $(k = 1, 2, 3, \ldots)$ be a sequence of convex continuous functionals in* H. *If this sequence is bounded at each point $h \in$ H, then the functional*

$$p(h) = \sup_n p_n(h)$$

is also convex and continuous.

Proof: That $p(h)$ is a convex functional is evident. On the other hand, for each $h_0 \in$ H and each $\varepsilon > 0$, there exists N such that

$$p(h_0) - p_N(h_0) < \frac{\varepsilon}{2}.$$

Then there exists $\delta > 0$ such that

$$|p_N(h) - p_N(h_0)| < \frac{\varepsilon}{2}$$

for $\|h - h_0\| < \delta$. But if $\|h - h_0\| < \delta$ then

$$p(h) - p(h_0) > \sup_n p_n(h) - p_N(h_0) - \frac{\varepsilon}{2} \geqq p_N(h) - p_N(h_0) - \frac{\varepsilon}{2} > -\varepsilon.$$

This implies that the functional $p(h)$ is lower semicontinuous. It remains only to apply the lemma, and the corollary is proved.

We give two simple applications of the propositions just proved. We know that each linear functional in $L^2(a, b)$ can be expressed in the form

$$\Phi(h) = \int_a^b h(t)\,\varphi(t)\,dt \tag{1}$$

where $\varphi(t)$ is the function in $L^2(a, b)$ which "represents" the functional $\Phi(h)$. We shall prove that if a functional $\Phi(h)$ is defined everywhere in $L^2(a, b)$ by means of formula (1), where $\varphi(t)$ is some fixed function, then this functional is necessarily linear, so that $\varphi(t)$ belongs to $L^2(a, b)$. In other words we shall prove that if the integral (1) exists for each function $h(t) \in L^2(a, b)$, then $\varphi(t) \in L^2(a, b)$. This fact is a special case of a more general theorem of F. Riesz.[6]

For the proof we denote by e_n the set of all points t which belong to the intersection of the intervals $[a, b]$, $[-n, n]$ and for which

$$|\varphi(t)| \leqq n.$$

Further let

$$p_n(h) = \int_{e_n} |h(t)\varphi(t)|\,dt.$$

This is a convex continuous functional in $L^2(a, b)$. The quantity

$$p(h) = \sup_n p_n(h) = \lim_{n\to\infty} p_n(h) = \int_a^b |h(t)\varphi(t)|\,dt$$

is finite for any $h(t) \in L^2(a, b)$. So, by the corollary of the lemma, the functional $p(h)$ is continuous, i.e., $p(h) \leqq M \| h \|$ for $h \in H$. But $|\Phi(h)| \leqq p(h)$ so that, since the homogeneity and additivity of the functional $\Phi(h)$ are evident, $\Phi(h)$ is a linear functional.

An analogous proposition is valid for the space l^2. We restrict ourselves to its formulation. Let a functional $\Phi(f)$ be defined everywhere in l^2 by means of the formula

$$\Phi(f) = \sum_{k=1}^{\infty} a_k x_k \qquad (f = \{x_k\}_1^\infty)$$

where $\{a_k\}_1^\infty$ is some fixed sequence. Then

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{2}$$

[6] Riesz's theorem pertains to the space $L^p(a, b)$ for any $p > 1$. (The space $L^p(a, b)$ is defined as the space of functions measurable in (a, b) for which $\int_a^b |f(x)|^p dx$ exists). See F. Riesz [1].

which implies that $\Phi(f)$ is a linear functional. In other words, if the series

$$\sum_{k=1}^{\infty} a_k x_k$$

converges for each sequence $\{x_k\}_1^{\infty}$ such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty$$

then the inequality (2) must hold. This fact is a special case of a more general theorem of E. Landau.[7]

19. Bounded Linear Operators

An operator T is *linear* if its domain of definition D is a linear manifold and if

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg$$

for any $f, g \in D$ and any complex numbers α and β.

We emphasize the fact that, in contrast with the definition of a linear functional, this definition does not require that the operator be bounded. This is related to the fact that many important operations of analysis such as, for instance, the operation of differentiation, generate unbounded but homogeneous and additive operators, i.e., operators which are linear in the sense of the definition given here.

A linear operator T is *bounded* if

$$\sup_{f \in D, \|f\| \leq 1} \|Tf\| < \infty.$$

The left member of this inequality is called the *norm* of the operator T in D and is denoted by the symbol $\|T\|$ or, sometimes, by $\|T\|_D$.

It is easy to see that the properties of Section 15 relating to linear functionals are also valid for bounded linear operators:

1. The norm of a bounded linear operator T can be defined equivalently by

$$\|T\| = \sup_{f \in D, \|f\| = 1} \|Tf\| = \sup_{f \in D} \frac{\|Tf\|}{\|f\|}.$$

2. A bounded linear operator is continuous.
3. If a linear operator is continuous at one point, then it is bounded.
4. The extension by continuity of a bounded linear operator T leads to a unique linear operator with the same norm as the original operator.

---

[7] Landau's theorem pertains to the space l^p for any $p > 1$. (The space l^p is the space of numerical sequences $x_1, x_2, \ldots$ for which the series $\sum_{i=1}^{\infty} |x_i|^p$ converges.) See E. Landau [1].

5. If S and T are linear operators, then $\alpha S + \beta T$, where α and β are complex numbers, is a linear operator with the intersection $D_S \cap D_T$ of the domains D_S and D_T as the domain of definition. Each of the products ST and TS (cf. Section 14) is also a linear operator. If S and T are bounded linear operators defined everywhere in H, then the operators ST and TS are also bounded linear operators defined everywhere in H, and

$$\|ST\| \leqq \|S\| \cdot \|T\|, \qquad \|TS\| \leqq \|T\| \cdot \|S\|.$$

20. Bilinear Functionals

We shall say that Ω is a bilinear functional defined in H,[8] if to each pair of elements $f, g \in H$ there corresponds a definite complex number $\Omega(f,g)$, and

(a) $\Omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Omega(f_1, g) + \alpha_2 \Omega(f_2, g)$,

(b) $\Omega(f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \Omega(f, g_1) + \bar{\beta}_2 \Omega(f, g_2)$,

(c) $\sup_{\|f\| \leqq 1, \|g\| \leqq 1} |\Omega(f,g)| < \infty$.

An example of a bilinear functional is the scalar product (f, g). The number

$$\sup_{\|f\| \leqq 1, \|g\| \leqq 1} |\Omega(f,g)|$$

is called the norm of the bilinear functional Ω, and is denoted by the symbol $\|\Omega\|$. It is not difficult to prove that

$$\|\Omega\| = \sup_{\|f\|=1, \|g\|=1} |\Omega(f,g)| = \sup \frac{|\Omega(f,g)|}{\|f\| \cdot \|g\|}.$$

Therefore, for any $f, g \in H$,

$$|\Omega(f,g)| \leqq \|\Omega\| \cdot \|f\| \cdot \|g\|.$$

A bilinear functional is a continuous function of each of its arguments, since

$$|\Omega(f,g) - \Omega(f_0,g_0)| = |\Omega(f-f_0, g-g_0) + \Omega(f-f_0, g_0) + \Omega(f_0, g-g_0)| \leqq$$
$$\leqq \|\Omega\| \{\|f-f_0\| \cdot \|g-g_0\| + \|f-f_0\| \cdot \|g_0\| + \|f_0\| \cdot \|g-g_0\|\}.$$

The following simple proposition is often useful.

THEOREM: *If a complex scalar function $\omega(f, g)$ satisfies the conditions*

(a) $\omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \omega(f_1, g) + \alpha_2 \omega(f_2, g)$,

(b) $\omega(f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \omega(f, g_1) + \bar{\beta}_2 \omega(f, g_2)$,

(c) $|\omega(f,f)| \leqq C \|f\|^2$,

(d) $|\omega(f,g)| = |\omega(g,f)|$,

[8] It is possible to introduce bilinear functionals which are not defined everywhere in H, but in what follows they will not be considered.

where C is a constant, f, f_1, f_2, g, g_1, g_2 are arbitrary elements of H *and* α_1, α_2, β_1, β_2 *are arbitrary complex numbers, then* ω *is a bilinear functional with norm* $\| \omega \| \leqq C$.

Proof: It is immediately proved by means of (a) and (b) that[9]

$$\omega(f,h) + \omega(h,f) = \tfrac{1}{2}\{\omega(f+h,f+h) - \omega(f-h,f-h)\}.$$

This implies that

$$(1)\quad |\omega(f,h)+\omega(h,f)| \leqq \tfrac{1}{2}C\{\|f+h\|^2+\|f-h\|^2\} = C\{\|f\|^2+\|h\|^2\}.$$

Let $\|f\| \leqq 1$, $\| h \| \leqq 1$ and $h = \lambda g$ where λ is a complex number such that $| \lambda | = 1$ (λ will be specified later). Then (1) yields

$$(2)\qquad |\bar{\lambda}\omega(f,g) + \lambda\omega(g,f)| \leqq 2C.$$

We suppose that $\omega(f, g) \neq 0$ and, in accordance with (d), let

$$\omega(f,g) = |\omega(f,g)|e^{i\alpha}, \qquad \omega(g,f) = |\omega(f,g)|e^{i\beta}.$$

Then, by (2),

$$|\omega(f,g)| \cdot |\bar{\lambda}e^{i\alpha} + \lambda e^{i\beta}| \leqq 2C.$$

Letting

$$\lambda = e^{i\frac{\alpha-\beta}{2}},$$

we find that

$$\bar{\lambda}e^{i\alpha} + \lambda e^{i\beta} = e^{i\frac{\alpha+\beta}{2}} + e^{i\frac{\alpha+\beta}{2}} = 2e^{i\frac{\alpha+\beta}{2}}$$

which yields

$$|\omega(f,g)| \leqq C \qquad (\|f\| \leqq 1, \|g\| \leqq 1).$$

This proves the theorem, since this relation is also correct for $\omega(f, g) = 0$.

COROLLARY: *If the bilinear functional* Ω *satisfies the condition*

$$|\Omega(f,g)| = |\Omega(g,f)|,$$

for $f, g \in$ H, *then*

$$\|\Omega\| = \sup_{f \in \mathrm{H}} \frac{|\Omega(f,f)|}{(f,f)}.$$

Proof: By the theorem,

$$\|\Omega\| \leqq \sup_{f \in \mathrm{H}} \frac{|\Omega(f,f)|}{(f,f)},$$

but on the other hand

$$\sup_{f \in \mathrm{H}} \frac{|\Omega(f,f)|}{(f,f)} \leqq \sup_{f,g \in \mathrm{H}} \frac{|\Omega(f,g)|}{\|f\| \cdot \|g\|} = \|\Omega\|.$$

[9] From this equation and the analogous equation

$$\omega(f,h) - \omega(h,f) = \frac{i}{2}\{\omega(f+ih,f+ih) - \omega(f-ih,f-ih)\}$$

it follows by means of (c) that ω is a bilinear functional with norm $\leqq 2C$. But, by means of condition (d), it is established that the norm of ω does not exceed C.

21. The General Form of a Bilinear Functional

THEOREM: *Each bilinear functional* $\Omega(f,g)$ *has a representation of the form*

$$\Omega(f,g) = (Af,g).$$

In this equation A *is a bounded linear operator with domain* H *which is uniquely determined by* Ω. *Furthermore,*

$$\|A\| = \|\Omega\|.$$

Proof: For fixed f, the expression $\overline{\Omega(f,g)}$ defines a linear functional in g with domain H. Consequently, according to the theorem of F. Riesz (cf. Section 16), there exists an element h_f, uniquely determined by the element f, for which

$$\overline{\Omega(f,g)} = (g,h_f)$$

or

$$\Omega(f,g) = (h_f,g)$$

for each $g \in \mathrm{H}$. Define the mapping A from H into H by the equation $Af = h_f$ for $f \in \mathrm{H}$. Then

$$\Omega(f,g) = (Af,g).$$

Since

$$\Omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Omega(f_1,g) + \alpha_2 \Omega(f_2,g)$$

we have

$$(A\{\alpha_1 f_1 + \alpha_2 f_2\} - \alpha_1 Af_1 - \alpha_2 Af_2, g) = 0$$

for $g \in \mathrm{H}$. Since g is arbitrary,

$$A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Af_1 + \alpha_2 Af_2,$$

so that A is a linear operator. The domain of the operator A is the whole space H. Furthermore, since

$$|(Af,g)| \leqq \|Af\| \cdot \|g\|$$

we have

$$\|\Omega\| = \sup \frac{|\Omega(f,g)|}{\|f\| \cdot \|g\|} = \sup \frac{|(Af,g)|}{\|f\| \cdot \|g\|} \leqq \sup \frac{\|Af\|}{\|f\|}.$$

On the other hand

$$\|\Omega\| = \sup \frac{|(Af,g)|}{\|f\| \cdot \|g\|} \geqq \sup \frac{(Af,Af)}{\|f\| \cdot \|Af\|} = \sup \frac{\|Af\|}{\|f\|}.$$

These relations show that the operator A is bounded and that

$$\|\Omega\| = \|A\|.$$

The operator A is uniquely determined by the linear functional Ω. In fact, if

$$\Omega(f,g) = (A'f,g) = (A''f,g),$$

for $f, g \in H$, then

$$(A'f - A''f, g) = 0.$$

But this is possible only for $A' = A''$.

22. Adjoint Operators

Let A be an arbitrary bounded linear operator defined on H. The expression

$$(f, Ag)$$

defines a bilinear functional on H with norm $\| A \|$. According to the theorem proved in the preceding section, there exists a unique bounded linear operator A^* defined on H with norm $\| A^* \| = \| A \|$ such that

$$(f, Ag) = (A^*f, g) \tag{1}$$

for $f, g \in H$. This operator A^* is called the *adjoint* of A. It is easy to see that the operator $(A^*)^* = A^{**}$ is equivalent to the original operator A.

If A is bounded and $A^* = A$, then A is said to be *self-adjoint*. A bounded linear operator A, defined on H, is said to be *normal* if it commutes with its adjoint, i.e., if

$$A^*A = AA^*.$$

Let A and B be two bounded linear operators defined on H. Then

$$(ABf, g) = (Bf, A^*g) = (f, B^*A^*g),$$

which implies that

$$(AB)^* = B^*A^*.$$

Therefore, the product of two self-adjoint operators is self-adjoint if and only if the operators commute.

THEOREM: *If A is a bounded self-adjoint operator, then*

$$\sup_{\|f\| = \|g\| = 1} |(Af, g)| = \sup_{\|f\| = 1} |(Af, f)|.$$

In other words,[10]

$$\|A\| = \max \{ |\Lambda|, |\lambda| \}$$

where

$$\Lambda = \sup_{\|f\| = 1} (Af, f), \quad \lambda = \inf_{\|f\| = 1} (Af, f).$$

Proof: The bilinear functional

$$\Omega(f, g) = (Af, g)$$

satisfies the condition

$$|\Omega(f, g)| = |\Omega(g, f)|.$$

Therefore, the corollary of the theorem of Section 20 applies and the theorem is proved.

[10] Translator's Note: It follows from (1) that (Af, f) is real for $f \in H$ if A is self-adjoint.

23. Weak Convergence in H

We say that the sequence of vectors $f_k \in H$, $(k = 1, 2, 3 \ldots)$ *converges weakly* to the vector f and we write $f_k \overset{w}{\to} f$ if

$$\lim_{k\to\infty}(f_k, h) = (f, h)$$

for $h \in H$. The concepts of a *weakly fundamental sequence* and of *weak completeness* are defined analogously.

If the sequence $\{f_k\}_1^\infty$ converges to f in the sense of Section 3, i.e., if

$$\lim_{k\to\infty} \|f_k - f\| = 0$$

then we shall continue to write $f_k \to f$, but we shall say, to avoid confusion, that the sequence converges *strongly* to f. Strong convergence implies weak convergence, but not conversely. Indeed, let $\{e_k\}_1^\infty$ be any infinite orthonormal sequence of vectors in H. Since, for any $h \in H$,

$$\sum_{k=1}^{\infty} |(h, e_k)|^2 \leqq (h, h)$$

(see Section 8), then for any $h \in H$,

$$\lim_{k\to\infty}(e_k, h) = 0.$$

Thus, the sequence $\{e_k\}_1^\infty$ converges weakly to the vector 0, but this sequence does not converge strongly since

$$\| e_k - e_i \|^2 = 2 \qquad (i \neq k)$$

so that $\|e_k - e_i\|$ does not converge to zero as $i, k \to \infty$. However, the following theorem is valid.

THEOREM 1: *If the sequence of vectors $\{f_k\}_1^\infty$ converges weakly to the vector f and if*

$$\lim_{k\to\infty} \|f_k\| = \|f\|,$$

then

$$\lim_{k\to\infty} \|f_k - f\| = 0,$$

i.e., the sequence $\{f_k\}_1^\infty$ converges strongly to the vector f.

Proof: The proof follows from the equation

$$\|f_k - f\|^2 = \|f_k\|^2 - (f_k, f) - (f, f_k) + \|f\|^2.$$

Indeed, by the hypothesis of the theorem,

$$\lim_{k\to\infty} \{\|f_k\|^2 - (f_k, f) - (f, f_k) + \|f\|^2\} = 0.$$

An important property of every weakly convergent sequence of vectors is boundedness. The proof of this property does not present any difficulty if the following general proposition is proved first.

THEOREM 2: *If the linear functionals* $\Phi_1, \Phi_2, \Phi_3, \ldots$, *defined on the space* H, *have the property that the numerical sequence* $\{\Phi_k(h)\}_1^\infty$ *is bounded for each* $h \in H$, *then the sequence* $\{\|\Phi_k\|\}_1^\infty$ *of the norms of the functionals is bounded.*

Proof: The proof follows almost immediately from the lemma concerning convex functionals proved in Section 18. For $h \in H$, let

$$p_n(h) = |\Phi_n(h)| \qquad (n = 1, 2, 3, \ldots).$$

The p_n are convex continuous functionals in H. By the lemma just mentioned, the convex functional

$$p(h) = \sup_n p_n(h)$$

is continuous, i.e.,

$$M = \sup_{\|h\| \leqq 1} p(h) < \infty.$$

Consequently,

$$\|\Phi_n\| \leqq M \qquad (n = 1, 2, 3, \ldots)$$

and the theorem is proved.

COROLLARY 1: *Every weakly convergent sequence* $\{f_k\}_1^\infty$ *is bounded.*

Proof: Each vector f_k determines a functional $\Phi_k(h) = (h, f_k)$. Since the sequence $\{f_k\}_1^\infty$ converges weakly, the numerical sequence $\{\Phi_k(h)\}_1^\infty$ converges for each $h \in H$ and, hence, is bounded. It remains only to apply Theorem 2 and to use the fact that

$$\|\Phi_k\| = \|f_k\|.$$

COROLLARY 2: *Every Hilbert space is weakly complete.*

Proof: Let the sequence of vectors $\{f_k\}_1^\infty$ be fundamental in the sense of weak convergence, i.e., for each $h \in H$, let

$$\lim_{m, n \to \infty} (f_n - f_m, h) = 0.$$

It follows that the sequence of numbers (f_k, h) $(k = 1, 2, 3, \ldots)$ converges for each fixed $h \in H$. According to Theorem 2 the sequence $\{f_k\}_1^\infty$ is bounded:

$$\|f_k\| \leqq M \qquad (k = 1, 2, 3, \ldots).$$

Therefore, the limit

$$\lim_{k \to \infty} (h, f_k)$$

defines a linear functional $\Phi(h)$ with norm $\leqq M$. According to the representation theorem of F. Riesz, $\Phi(h) = (h, f)$, where f is a unique element of the space H. This element is the weak limit of the sequence $\{f_k\}_1^\infty$.

24. Weak Compactness

A point set is said to be *compact*[11] if every sequence belonging to it contains a convergent subsequence. Corresponding to the two types of convergence (strong and weak) are *strong* (or *ordinary*) *compactness* and *weak compactness*. The concept of compactness is associated with the most important theorem of elementary analysis — the Bolzano-Weierstrass theorem. The conclusion of this theorem is false for a Hilbert space if the theorem is stated in terms of strong convergence. To prove this, it is sufficient to take the infinite orthonormal sequence of vectors $e_1, e_2, e_3, \ldots$. This sequence is bounded, but none of its subsequences is strongly convergent. In connection with what has been said, it may be surprising that the following theorem holds.

THEOREM 1: *Every bounded point set in* H *is weakly compact.*

Proof: Let us take any sequence $\{g_k\}_1^\infty$ of points such that, for some C,

$$\|g_k\| \leqq C < \infty \qquad (k = 1, 2, 3, \ldots).$$

Let L denote the linear envelope of the set $\{g_k\}_1^\infty$, and let $G = \bar{L}$ be its closure. Define F by

$$F = H \ominus G.$$

Consider the numerical sequence

$$(1) \qquad (g_1, g_k), \qquad (k = 1, 2, 3, \ldots).$$

It is bounded because

$$|(g_1, g_k)| \leqq \|g_1\| \cdot \|g_k\| \leqq C^2 \qquad (k = 1, 2, 3, \ldots).$$

Therefore, the sequence (1) contains a convergent subsequence. In other words, $\{g_k\}_1^\infty$ contains a subsequence $\{g_{1k}\}_{k=1}^\infty$ for which

$$\lim_{k\to\infty} (g_1, g_{1k})$$

exists. Similarily, from the boundedness of the numerical sequence

$$(2) \qquad (g_2, g_{1k})$$

we conclude that $\{g_{1k}\}_{k=1}^\infty$ contains a subsequence $\{g_{2k}\}_{k=1}^\infty$ for which

$$\lim_{k\to\infty} (g_2, g_{2k})$$

exists. Repeating this argument, we get an infinite sequence of sequences

$$\begin{aligned} &g_{11}, g_{12}, g_{13}, \ldots, \\ &g_{21}, g_{22}, g_{23}, \ldots, \\ &g_{31}, g_{32}, g_{33}, \ldots, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

each of which is a subsequence of the preceding. It is evident that the diagonal sequence

[11] Translator's Note: This general concept is often called *sequential compactness.*

$$g_{11}, g_{22}, g_{33}, \ldots$$

has the property that, for each integer r

$$\lim_{k\to\infty} (g_r, g_{kk})$$

exists. Hence, it follows that

$$\lim_{k\to\infty} (g, g_{kk})$$

exists for each $g \in L$ and, therefore, for each $g \in G$. If $f \in F$, then

$$(f, g_{kk}) = 0, \qquad (k = 1, 2, 3, \ldots).$$

Consequently,

$$\lim_{k\to\infty} (f, g_{kk})$$

exists for each $f \in F$. Since $H = F \oplus G$, the results we have obtained imply the existence, for each $h \in H$, of

$$\lim_{k\to\infty} (h, g_{kk}).$$

Therefore, the sequence $\{g_{kk}\}_{k=1}^{\infty}$ is fundamental in the sense of weak convergence. By the weak completeness of the space, this sequence converges weakly to some element of H, and this proves our theorem.

THEOREM 2: *For the weak convergence of the sequence of vectors* $\{g_k\}_1^{\infty}$ *it is necessary and sufficient that:*

1. *the numerical sequence*

$$(g_k, f) \qquad (k = 1, 2, 3, \ldots)$$

converge for each f *of some set* M *which is dense in* H; *and*

2. *the sequence* $\{g_k\}_{k=1}^{\infty}$ *be bounded, i.e, the inequality*

$$\|g_k\| \leqq C < \infty \qquad (k = 1, 2, 3, \ldots)$$

hold for some C.

Proof: The necessity of condition 1 is evident. The necessity of condition 2 is indicated by Corollary 1 of Theorem 2 of Section 23. We turn to the proof of the sufficiency of the conditions mentioned. By Theorem 1 of the preceding paragraph, $\{g_k\}_{k=1}^{\infty}$ has a weakly convergent subsequence $\{g_{k_i}\}_{i=1}^{\infty}$. Let g be the weak limit of this subsequence.
Then

$$\lim_{i\to\infty} (h, g_{k_i}) = (h, g).$$

According to condition 1 of the theorem,

$$\lim_{k\to\infty} (f, g_k)$$

exists for each $f \in M$. Therefore,

$$\lim_{k\to\infty} (f, g_k) = (f, g)$$

for $f \in M$, and it remains to prove (we leave this to the reader) that this equation holds if f is any element of H.

25. A Criterion for the Boundedness of an Operator

THEOREM: *Let A and A^* be linear operators defined on* H *and assume that*

$$(Af, g) = (f, A^*g)$$

for $f, g \in$ H. Then A is bounded, and A^* is the adjoint of A.

Proof: We assume the contrary and suppose that there exists a sequence of vectors $\{f_k\}_1^\infty$ such that

$$\|f_k\| = 1, \quad \|Af_k\| > k \qquad (k = 1, 2, 3, \ldots).$$

The expressions

$$(g, Af_k) = \Phi_k(g) \qquad (k = 1, 2, 3, \ldots)$$

define linear functionals Φ_k in H. Since

$$\Phi_k(g) = (A^*g, f_k) \qquad (k = 1, 2, 3, \ldots),$$

the numerical sequence $\{\Phi_k(g)\}_{k=1}^\infty$ is bounded. By Theorem 2 of Section 23, the sequence of norms $\|\Phi_k\|$ $(k = 1, 2, 3, \ldots)$, i.e., the sequence of numbers $\|Af_k\|$, is also bounded, which is a contradiction. Thus, the theorem is proved.

An important special case of this theorem is due to Hellinger and Toeplitz. We mention it in the following section.

26. Linear Operators in a Separable Space

In this section we shall consider linear operators defined everywhere on a separable Hilbert space H. We show that bounded operators admit matrix representations completely analogous to the well known matrix representations of operators on finite dimensional spaces.

We choose any orthonormal basis $\{e_k\}_1^\infty$ in H and let[12]

$$Ae_k = c_k, \qquad (k = 1, 2, 3, \ldots)$$

and

$$(Ae_k, e_i) = a_{ik} \qquad (i, k = 1, 2, 3, \ldots). \tag{1}$$

Thus

$$c_k = \sum_{i=1}^{\infty} a_{ik} e_i \qquad (k = 1, 2, 3, \ldots).$$

Moreover,

$$\sum_{i=1}^{\infty} |a_{ik}|^2 < \infty \qquad (k = 1, 2, 3, \ldots).$$

[12] We remark that if the operator A is not defined everywhere in H, but only on a set D which is dense in H, then there exists in H an orthonormal basis $\{e_k\}_1^\infty$, the elements of which belong to D.

We introduce the infinite matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

of which the elements of the kth column are the components of the vector into which the operator A maps the kth coordinate vector. If the operator A is bounded, then it is uniquely determined by the infinite matrix (a_{ik}). For the proof of this assertion it is necessary to show how to represent the operator in terms of the matrix and the orthonormal basis $\{e_k\}_1^\infty$. First, we have

$$Ae_k = \sum_{i=1}^{\infty} a_{ik} e_i \qquad (k = 1, 2, 3, \dots).$$

Since the operator A is linear, it is well defined on the linear envelope of the given basis, i.e., for all vectors each of which has only a finite number of nonzero components relative to the basis. Since A is continuous, the value of Af for an arbitrary vector $f \in H$ may be found by means of a passage to a limit.

It is not difficult to write a simple formula for the components of the vector f; indeed, if

$$f = \sum_{k=1}^{\infty} x_k e_k \tag{2}$$

then

$$Af = \sum_{k=1}^{\infty} y_k e_k, \tag{3}$$

where

$$y_k = \sum_{i=1}^{\infty} a_{ki} x_i. \tag{4}$$

In fact, if

$$f_n = \sum_{k=1}^{n} x_k e_k,$$

then

$$Af_n = \sum_{k=1}^{\infty} y_k^{(n)} e_k$$

where

$$y_k^{(n)} = \sum_{i=1}^{n} a_{ki} x_i.$$

By the boundedness of the operator A,

$$y_k = (Af, e_k) = \lim_{n\to\infty} (Af_n, e_k) = \lim_{n\to\infty} y_k^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{n} a_{ki} x_i = \sum_{i=1}^{\infty} a_{ki} x_i.$$

DEFINITION: *If the operator A is defined everywhere in* H *and if its value for any vector* (2) *is given by the formulas* (3) *and* (4), *then we say that the operator A admits a matrix representation relative to the orthogonal basis* $\{e_k\}_1^\infty$.

Thus, we have proved that every bounded linear operator defined on the entire space admits a matrix representation with respect to each orthogonal basis. This is the analogue, mentioned at the very beginning of the present section, between a separable Hilbert space and a finite-dimensional space, with respect to bounded linear operators.

THEOREM: *If an operator A, defined everywhere in a separable space* H, *admits a matrix representation with respect to some orthogonal basis, then it is bounded.*

(This proposition is a frequently used special case of the theorem of the preceding section, mentioned above, which is due to Hellinger and Toeplitz).[13]

Proof: By hypothesis, the series

$$(Af, e_k) = \sum_{i=1}^{\infty} a_{ki} x_i \qquad (k = 1, 2, 3, \ldots)$$

converges for each vector

$$f = \sum_{k=1}^{\infty} x_k e_k,$$

where $\{e_k\}_1^\infty$ is the orthonormal basis, mentioned in the theorem, with respect to which the operator A admits a matrix representation. Therefore, by the theorem of Landau (see Section 18),

$$\sum_{i=1}^{\infty} |a_{ki}|^2 < \infty \qquad (k = 1, 2, 3, \ldots). \tag{5}$$

We introduce the sequence of vectors

$$c_k^* = \sum_{i=1}^{\infty} \bar{a}_{ki} e_i \qquad (k = 1, 2, 3, \ldots)$$

and by means of them, define the linear operator A^*. First, let

$$A^* e_k = c_k^* \qquad (k = 1, 2, 3, \ldots)$$

and then use linearity to define A^* on the linear envelope of the set of vectors e_k. Finally, extend A^* by continuity to all of H. It is easy to prove that for any $f, g \in \mathrm{H}$,

$$(Af, g) = (f, A^* g)$$

after which, to complete the proof, it remains to apply the theorem of the preceding section.

[13] E. Hellinger and O. Toeplitz [1].

We shall not present all the details of the proof just outlined, but we mention another proof of the theorem, which is based directly on the lemma concerning convex functionals in Section 18. In view of inequality (5), the expression

$$\Phi_k(f) = \sum_{i=1}^{\infty} a_{ki} x_i \qquad (k = 1, 2, 3, \ldots)$$

defines a linear functional of

$$f = \sum_{k=1}^{\infty} x_k e_k.$$

Therefore,

$$p_n(f) = \sqrt{\sum_{k=1}^{n} |\Phi_k(f)|^2} \qquad (n = 1, 2, 3, \ldots)$$

defines a convex continuous functional of f. Since

$$\sum_{k=1}^{n} |\Phi_k(f)|^2 = \sum_{k=1}^{n} |(Af, e_k)|^2 \leqq \|Af\|^2$$

the sequence $\{p_n(f)\}_{n=1}^{\infty}$ is bounded for each $f \in H$. On the basis of the corollary of the lemma concerning convex functionals, the functional

$$p(f) = \sup_n p_n(f) = \lim_{n\to\infty} p_n(f) = \sqrt{\sum_{k=1}^{\infty} |\Phi_k(f)|^2} = \|Af\|$$

is continuous, i.e., there exists a constant M such that

$$p(f) \leqq M\|f\|.$$

But this implies that the operator A is bounded.

The proof of the theorem can be formulated also in the following form: if for arbitrary numbers $x_k (k = 1, 2, 3, \ldots)$ such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty$$

the inequality

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 < \infty$$

holds, then there exists a constant M such that

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 \leqq M^2 \sum_{k=1}^{\infty} |x_k|^2.$$

This reduces to the theorem of E. Landau (see Section 18) if $a_{ki} = 0$ for $k > 1$.

Let us agree to write

$$A \sim (a_{ik})$$

if the bounded linear operator A, defined everywhere in H, corresponds to the matrix (a_{ik}) according to (1). Here the orthogonal basis $\{e_k\}_1^\infty$ is arbitrary but fixed.

If

$$A \sim (a_{ik}), \qquad B \sim (b_{ik})$$

then, as is easily verified,

$$AB \sim (c_{ik})$$

where

$$c_{ik} = \sum_{r=1}^{\infty} a_{ir} b_{rk} \qquad (i, k = 1, 2, 3, \ldots).$$

If we define matrix multiplication by means of this equation, then

$$AB \sim (a_{ir}) \cdot (b_{rk}).$$

Furthermore, if

$$A \sim (a_{ik})$$

and

$$A^* \sim (a_{ik}^*)$$

then

$$a_{ik}^* = \bar{a}_{ki} \qquad (i, k = 1, 2, 3, \ldots).$$

Therefore, the condition that the bounded operator A be self-adjoint may be expressed in the form

$$a_{ik} = \bar{a}_{ki}. \tag{6}$$

Matrices for which equation (6) holds are called *symmetric* or *Hermitean.*

A bilinear functional is generated by the operator A by means of the equation

$$(Af, g) = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} a_{ki} x_i \Big) \bar{y}_k.$$

In this equation

$$f = \sum_{k=1}^{\infty} x_k e_k, \qquad g = \sum_{k=1}^{\infty} y_k e_k.$$

In the double sum appearing above it is possible to reverse the order of summation, since the equation

$$(Af, g) = (f, A^* g)$$

implies that

$$\sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} a_{ki} x_i \Big) \bar{y}_k = \sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} a_{ki} \bar{y}_k \Big) x_i.$$

From the inequality

$$|(Af, g)| \leqq M \|f\| \cdot \|g\| \tag{7}$$

it follows that

$$\left| \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ki} x_i \bar{y}_k \right| \leqq M \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{k=1}^{\infty} |y_k|^2}.$$

If each of the vectors f and g has only a finite number of nonzero components, then the last inequality may be written in the form

$$\left|\sum_{i=1}^{p}\sum_{k=1}^{q}a_{ki}x_i\bar{y}_k\right| \leqq M\sqrt{\sum_{i=1}^{p}|x_i|^2}\sqrt{\sum_{k=1}^{q}|y_k|^2}. \tag{8}$$

THEOREM: *In order that the matrix* (a_{ik}) *represent a bounded linear operator defined everywhere in* H, *it is necessary and sufficient that, for some constant* M, *the inequality* (8) *hold for any numbers* $x_1, x_2, \ldots, x_p$ *and* $y_1, y_2, \ldots, y_q$.

Proof: If the operator A is bounded and

$$a_{ik} = (Ae_k, e_i) \qquad (i, k = 1, 2, 3, \ldots),$$

then (7) implies (8). Now let (a_{ik}) be a matrix which satisfies condition (8). We shall show that this matrix determines a bounded linear operator A. First, from (8) with

$$x_1 = x_2 = \ldots = x_{k-1} = x_{k+1} = \ldots = 0, x_k \neq 0,$$
$$y_1 = y_2 = \ldots = y_{m-1} = y_{n+1} = y_{n+2} = \ldots = 0$$

we get

$$\left|\sum_{i=m}^{n}a_{ik}\bar{y}_i\right| \leqq M\sqrt{\sum_{i=m}^{n}|y_i|^2}.$$

This implies the convergence of the series

$$\sum_{i=1}^{\infty}a_{ik}\bar{y}_i$$

for any sequence $\{y_i\}_1^\infty$ in l^2. Hence, the theorem of Landau (see Section 18) implies the convergence of the series

$$\sum_{i=1}^{\infty}|a_{ik}|^2 \qquad (k = 1, 2, 3, \ldots).$$

We define the operator A_0, first for the basis elements by the formula

$$A_0e_k = \sum_{i=1}^{\infty}a_{ik}e_i \qquad (k = 1, 2, 3, \ldots),$$

and then, by means of linearity, for all vectors with only a finite number of components different from zero. Now we prove that the operator A_0 is bounded. We have, by (8) for f and g with only finite number of nonzero components,

$$|(A_0f, g)| \leqq M\|f\| \cdot \|g\|. \tag{9}$$

By the continuity of the scalar product, the inequality (9) is satisfied for all $g \in$ H. Let

$$g = A_0f$$

in (9) to get

$$\|A_0f\|^2 \leqq M\|f\| \cdot \|A_0f\|$$

so that

$$\| A_0 f \| \leqq M \| f \|.$$

Thus, A_0 is bounded. Extending A_0 by continuity to the whole space **H**, we get the bounded operator A and the correspondence

$$A \sim (a_{ik}).$$

The theorem is proved.

We note that if the matrix (a_{ik}) is symmetric (Hermitean), i.e., if

$$a_{ik} = \bar{a}_{ki}$$

then it is possible to replace the condition (8) by (see Section 22)

$$\left| \sum_{i,k=1}^{p} a_{ik} x_i \bar{x}_k \right| \leqq M \sum_{i=1}^{p} |x_i|^2.$$

We now give an example of the matrix representation of a bounded linear operator. Consider the *integral operator* in $L^2(-\infty, \infty)$ defined by the formula

$$g(s) = Af(s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt$$

where the function $K(s, t)$ is called the *kernel* of the operator. If the kernel satisfies the condition

$$\int_{-\infty}^{\infty}\int |K(s,t)|^2 ds\, dt < \infty, \tag{10}$$

it is called a *Hilbert-Schmidt kernel*, and the operator determined by it is called a *Hilbert-Schmidt operator*. We suppose that the condition (10) is satisfied. Then, for almost all t and u

$$\int_{-\infty}^{\infty} |K(s,t) \cdot \overline{K(s,u)}|\, ds \leqq \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds} \sqrt{\int_{-\infty}^{\infty} |K(s,u)|^2 ds}\,.$$

But since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)| \cdot |f(u)| \cdot \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds} \sqrt{\int_{-\infty}^{\infty} |K(s,u)|^2 ds\, dt\, du} =$$

$$= \left(\int_{-\infty}^{\infty} |f(t)| \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt} \right)^2 \leqq$$

$$\leqq \int_{-\infty}^{\infty} |f(t)|^2 dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt < \infty,$$

we have, by Fubini's theorem,

$$\| g \| = \left\{ \int_{-\infty}^{\infty} ds \left| \int_{-\infty}^{\infty} K(s,t) f(t)\, dt \right|^2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} K(s,t) f(t)\, dt \int_{-\infty}^{\infty} \overline{K(s,u)}\, \overline{f(u)}\, du \right\}^{\frac{1}{2}} \leqq$$

$$\leqq \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt} \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt}\,.$$

We see that the Hilbert-Schmidt operator is bounded and that its norm does not exceed the quantity

$$\sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt}.$$

Let us take in $L^2(-\infty, \infty)$ any complete orthonormal system of functions $\{\varphi_k(t)\}_1^\infty$ and define

$$a_{ik} = \int_{-\infty}^{\infty}\int K(s,t)\, \overline{\varphi_i(s)}\, \varphi_k(t)\, ds\, dt \qquad (i,k = 1,2,3,\ldots).$$

Choose any $f(t) \in L^2(-\infty, \infty)$ and let

$$\int_{-\infty}^{\infty} f(t)\, \overline{\varphi_k(t)}\, dt = x_k \qquad (k = 1,2,3,\ldots).$$

Then the Fourier coefficients

$$y_i = \int_{-\infty}^{\infty} g(s)\, \overline{\varphi_i(s)}\, ds \qquad (i = 1,2,3,\ldots)$$

of the function

$$g(s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt$$

are given by

$$y_i = \int_{-\infty}^{\infty}\int K(s,t) f(t)\, \overline{\varphi_i(s)}\, ds\, dt =$$

$$= \int_{-\infty}^{\infty} f(t) \left\{ \int_{-\infty}^{\infty} K(s,t)\, \overline{\varphi_i(s)}\, ds \right\} dt \qquad (i = 1,2,3,\ldots).$$

But since

$$(11) \qquad \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \sim \sum_{k=1}^{\infty} a_{ik}\,\overline{\varphi_k(t)} \qquad (i = 1, 2, 3, \ldots)$$

and

$$f(t) \sim \sum_{k=1}^{\infty} x_k\,\varphi_k(t),$$

we have, by the Parseval relation,

$$y_i = \sum_{k=1}^{\infty} a_{ik}\,x_k \qquad (i = 1, 2, 3, \ldots).$$

In a similar way, it follows from relation (11) that

$$\int_{-\infty}^{\infty} dt \left| \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \right|^2 = \sum_{k=1}^{\infty} |a_{ik}|^2 \qquad (i = 1, 2, 3, \ldots).$$

But on the other hand, by the Parseval relation,

$$\int_{-\infty}^{\infty} |K(s,t)|^2\,ds = \sum_{i=1}^{\infty} \left| \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \right|^2$$

and, therefore,

$$\int_{-\infty}^{\infty}\int |K(s,t)|^2\,ds\,dt = \sum_{i,k=1}^{\infty} |a_{ik}|^2.$$

We see that each Hilbert-Schmidt operator is represented by a matrix operator for which

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty.$$

27. Completely Continuous Operators

Hilbert considered first the important class of completely continuous operators. A linear operator A defined everywhere in H is said to be *completely continuous* if it maps each bounded point set into a set which is compact in the sense of strong convergence.

A completely continuous operator A is bounded. In fact, otherwise there would exist a sequence of points f_k $(k = 1, 2, 3, \ldots)$ for which

$$\|f_k\| = 1, \quad \|Af_k\| > k \qquad (k = 1, 2, 3, \ldots)$$

but this is impossible, since the point set $\{Af_k\}_1^{\infty}$ is compact.

Completely continuous operators have another definition: a linear operator A defined everywhere in H is completely continuous if it maps every weakly convergent sequence into a strongly convergent sequence. The proof of the equivalence of these definitions we leave to the reader.

We also leave to the reader the proofs of the following simple facts:

1. If A is a completely continuous operator and if B is a bounded operator defined everywhere in H, then the operators AB and BA are completely continuous.

2. If A_1 and A_2 are completely continuous operators, then $\alpha_1 A_1 + \alpha_2 A_2$ is a completely continuous operator.

THEOREM: *If A is a bounded linear operator defined everywhere in* H, *and if the operator A^*A is completely continuous, then the operator A is completely continuous.*

Proof: Let M be any bounded infinite set of points f ($\|f\| \leqq C$). Let $\{f_k\}_1^\infty$ be any sequence of elements of this set. This sequence is mapped by the operator A^*A into a strongly convergent sequence. Since

$$\|Af_n - Af_m\|^2 = (A(f_n - f_m), A(f_n - f_m)) =$$
$$= (A^*A(f_n - f_m), f_n - f_m) \leqq \|A^*Af_n - A^*Af_m\| \cdot \|f_n - f_m\|,$$

we have

$$\lim_{m,\, n \to \infty} \|A^*Af_n - A^*Af_m\| = 0$$

where

$$\|f_n - f_m\| \leqq 2C.$$

Therefore,

$$\lim_{m,\, n \to \infty} \|Af_n - Af_m\| = 0,$$

so that the sequence $\{Af_n\}_1^\infty$ converges and the theorem is proved.

COROLLARY: *If the operator A is completely continuous, then the operator A^* has the same property.*

Proof: In fact, if the operator A is completely continuous, then the operator $AA^* = (A^*)^*A^*$ is completely continuous, and it remains only to apply the theorem just proved.

28. A Criterion for Complete Continuity of an Operator

The following theorem is often used to prove that a given operator is completely continuous.

THEOREM: *If for each $\varepsilon > 0$ there exists a completely continuous operator A_ε such that*

$$\|(A - A_\varepsilon)f\| \leqq \varepsilon \|f\|$$

for $f \in$ H then the operator A is completely continuous.

Proof: We choose a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \ldots$ ($\lim_{n \to \infty} \varepsilon_n = 0$) and consider a sequence of completely continuous operators $A_{\varepsilon_1}, A_{\varepsilon_2}, \ldots$ corresponding to it by the condition of the theorem. Let M be an arbitrary bounded set of points $f(\|f\| \leqq C)$ in the space H. Let us

take an arbitrary sequence $\{f_k\}_1^\infty$ of points belonging to M. According to the complete continuity of A_{ε_1} there exists a subsequence

$$f_{11}, f_{12}, f_{13}, \ldots \tag{1}$$

which is mapped by the operator A_{ε_1} into a convergent sequence. From the sequence (1) we select a subsequence

$$f_{21}, f_{22}, f_{23}, \ldots \tag{2}$$

which is mapped into a convergent sequence by the operator A_{ε_2}. Continuing this process, we get the infinite sequence of sequences

$$\begin{aligned} &f_{11}, f_{12}, f_{13}, \ldots, \\ &f_{21}, f_{22}, f_{23}, \ldots, \\ &f_{31}, f_{32}, f_{33}, \ldots, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

such that each is a subsequence of the preceding. The diagonal sequence

$$f_{11}, f_{22}, f_{33}, \ldots$$

is mapped into a strongly convergent sequence by each of the operators A_{ε_i}. We prove next that the diagonal sequence $\{f_{kk}\}_{k=1}^\infty$ is mapped into a convergent sequence also by the operator A. For this it suffices to prove that

$$\lim_{m,\, n\to\infty} \|Af_{nn} - Af_{mm}\| = 0. \tag{3}$$

We have the inequality

$$\begin{aligned} \|Af_{nn} - Af_{mm}\| &\leqq \|(A - A_{\varepsilon_k})f_{nn}\| + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\| + \\ &+ \|(A - A_{\varepsilon_k})f_{mm}\| \leqq 2\varepsilon_k C + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\|. \end{aligned}$$

By taking k sufficiently large, we can make the term $2\varepsilon_k C$ as small as desired. After this, we can take N so large that the second term of the last member is made as small as desired for $m, n > N$. Thus, the relation (3) is proved.

We make use of the theorem just proved in order to establish the complete continuity of the matrix operator defined by

$$y_i = \sum_{k=1}^{\infty} a_{ik}x_k \qquad (i = 1, 2, 3 \ldots),$$

where

$$\sum_{i,\, k=1}^{\infty} |a_{ik}|^2 < \infty, \tag{4}$$

which implies the complete continuity of every integral operator with a Hilbert-Schmidt kernel. From (4) it follows that

$$\lim_{p\to\infty} \sum_{i=1}^{p} \sum_{k=1}^{\infty} |a_{ik}|^2 < \infty.$$

Therefore, for each $\varepsilon > 0$ there exists an integer $p = p(\varepsilon)$ such that

$$\sum_{i=p+1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 \leqq \varepsilon^2.$$

Now we construct the operator A_ε with the aid of the relation

$$A_\varepsilon f = y_1 e_1 + y_2 e_2 + \ldots + y_p e_p$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k \qquad (i = 1, 2, 3, \ldots)$$

if

$$f = \sum_{k=1}^{\infty} x_k e_k.$$

Let

$$Af = \sum_{i=1}^{\infty} y_i e_i.$$

We have

$$\|Af - A_\varepsilon f\|^2 = \sum_{i=p+1}^{\infty} |y_i|^2 =$$

$$= \sum_{i=p+1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} x_k \right|^2 \leqq \sum_{i=p+1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 \|f\|^2 \leqq \varepsilon^2 \|f\|^2.$$

It remains only to verify that the operator A_ε is completely continuous. Choose any bounded set of vectors in H. The operator A_ε maps this set into a bounded set in a finite-dimensional subspace of H, and this set is compact by the classical Bolzano-Weierstrass theorem.

We emphasize the fact that the convergence of the series

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2$$

is only a sufficient, but not a necessary condition for the complete continuity of the matrix operator. In the special case when the numbers a_{ik} satisfy the relation

$$a_{ik} = 0 \text{ for } |i - k| > r \qquad (i, k = 1, 2, 3, \ldots)$$

for some fixed r, it is possible to specify a necessary and sufficient condition for complete continuity. It is expressed by the relation

$$\lim_{i,k \to \infty} a_{ik} = 0.$$

For simplicity we sketch the proof only in the case with $r = 1$. In this case, the matrix defining the operator has the form

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & \dots \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & 0 & \dots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 & \dots \\ 0 & 0 & \gamma_3 & \alpha_4 & \beta_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{5}$$

and is called a *Jacobi matrix*. For $r > 1$ the matrix is called a *generalized Jacobi matrix*. Let the operator A, determined by the matrix (5), be completely continuous. Then the sequence of vectors

$$Ae_i = \beta_{i-1} e_{i-1} + \alpha_i e_i + \gamma_i e_{i+1}$$
$$(\beta_0 = 0, \quad i = 1, 2, 3, \dots)$$

must converge strongly. Supposing that the matrix A does not satisfy the condition in question, we select a sequence $i_1, i_2, i_3, \dots$ such that

$$i_k \geqq i_{k-1} + 3$$

and

$$|\beta_{i_k - 1}|^2 + |\alpha_{i_k}|^2 + |\gamma_{i_k}|^2 \to \delta > 0$$

where $\delta \leqq \infty$. A simple computation yields

$$\|Ae_{i_n} - Ae_{i_m}\|^2 = |\beta_{i_n - 1}|^2 + |\alpha_{i_n}|^2 + |\gamma_{i_n}|^2 + \\ + |\beta_{i_m - 1}|^2 + |\alpha_{i_m}|^2 + |\gamma_{i_m}|^2 \to 2\delta \neq 0.$$

This contradicts the strong convergence of the sequence $\{Ae_k\}_1^\infty$.

We now prove the sufficiency of the assertion. Let

$$\alpha_k \to 0, \ \beta_k \to 0, \ \gamma_k \to 0 \qquad (k \to \infty)$$

and let the sequence $\{f^{(k)}\}_1^\infty$ converge weakly to f. Because

$$Af^{(n)} = \sum_{k=1}^{\infty} x_k^{(n)} Ae_k = \sum_{k=1}^{\infty} x_k^{(n)} (\beta_{k-1} e_{k-1} + \alpha_k e_k + \gamma_k e_{k+1}) =$$

$$= \sum_{k=1}^{\infty} (\beta_k x_{k+1}^{(n)} + \alpha_k x_k^{(n)} + \gamma_{k-1} x_{k-1}^{(n)}) e_k \qquad (\gamma_0 = 0)$$

we have

$$\|Af^{(n)} - Af^{(m)}\|^2 =$$

$$= \sum_{k=1}^{\infty} |\beta_k \{x_{k+1}^{(n)} - x_{k+1}^{(m)}\} + \alpha_k \{x_k^{(n)} - x_k^{(m)}\} + \gamma_{k-1} \{x_{k-1}^{(n)} - x_{k-1}^{(m)}\}|^2$$

$$= \sum_{k=1}^{q} + \sum_{k=q+1}^{\infty}.$$

The first term on the right side tends toward zero for fixed q as $m, n \to \infty$. Therefore, it is sufficient to show that it is possible to make the second term on the right side as small as desired for all m and n by taking q sufficiently large. But, if q is sufficiently large and $k > q$, then

$$|\beta_k| < \varepsilon, |\alpha_k| < \varepsilon, |\gamma_{k-1}| < \varepsilon.$$

Therefore,

$$\sum_{k=q+1}^{\infty} |\beta_k\{x_{k+1}^{(n)} - x_{k+1}^{(m)}\} + \alpha_k\{x_k^{(n)} - x_k^{(m)}\} + \gamma_{k-1}\{x_{k-1}^{(n)} - x_{k-1}^{(m)}\}|^2 \leqq$$
$$\leqq 9\varepsilon^2 \|f^{(n)} - f^{(m)}\|^2.$$

Thus, our assertion is proved.

29. Sequences of Bounded Linear Operators

We distinguish three modes of convergence of a sequence $\{A_n\}_1^\infty$ of bounded linear operators defined everywhere in H: *weak* convergence, *strong* convergence (or, simply, convergence), and *uniform* convergence. A sequence $\{A_n\}_1^\infty$

converges weakly to the operator A $A_n \xrightarrow{w} A$	if for each $f \in H$ $A_n f \xrightarrow{w} Af \quad (n \to \infty)$
converges strongly to the operator A $A_n \to A$	if for each $f \in H$ $A_n f \to Af \quad (n \to \infty)$
converges uniformly to the operator A $A_n \Rightarrow A$	if $\|A_n - A\| \to 0 \quad (n \to \infty)$

If a sequence of operators converges uniformly, then it also converges strongly; if it converges strongly, then it also converges weakly.

Using the results of Section 23 and the lemma about convex functionals in Section 18, it is possible to prove the following proposition: if the sequence $\{A_n\}_1^\infty$ of bounded linear operators defined everywhere in H converges weakly, then the sequence $\{\|A_n\|\}_1^\infty$ of the norms of these operators is bounded.

We mention one more proposition, which is analogous to Corollary 2 in Section 23: if the sequence of bilinear functionals $\{\Omega_n(f, g)\}_1^\infty$ has the property that for arbitrary f and g the limit

$$\lim_{n \to \infty} \Omega_n(f, g) = \omega(f, g)$$

exists and is finite, then this limit defines a bilinear functional. As is easily seen, it is sufficient to prove that

$$|\omega(f,g)| \leqq C < \infty$$

for $\|f\| \leqq 1$, $\|g\| \leqq 1$, and some number C. Each of the bilinear functionals $\Omega_n(f, g)$ is determined by a particular bounded linear operator:

$$\Omega_n(f, g) = (A_n f, g).$$

The hypothesis implies that the sequence of operators $\{A_n\}_1^\infty$ converges weakly. Consequently, for $f, g \in \mathrm{H}$,

$$|(A_n f, g)| \leqq C \|f\| \cdot \|g\|$$

or equivalently,

$$|\Omega_n(f,g)| \leqq C \|f\| \cdot \|g\|.$$

Hence, it follows that

$$|\omega(f,g)| \leqq C \|f\| \cdot \|g\|.$$

Chapter III

PROJECTION OPERATORS AND UNITARY OPERATORS

30. Definition of a Projection Operator

Let G be a subspace of the space H and let

$$F = H \ominus G,$$

so that

$$H = G \oplus F.$$

Then each vector $h \in H$ is uniquely representable in the form

$$h = g + f,$$

where $g \in G$ and $f \in F$. In Section 7 the vector g was called the projection of h on G. The operator which maps each $h \in H$ into its projection g on G is called the operator of projection on G or, simply, a projection operator. It is denoted by P_G or sometimes, when the subspace G is specified in advance, by P. Thus, if g and h are related as above,

$$g = Ph = P_G h.$$

A projection operator is evidently linear. In addition, it is bounded and its norm is equal to one. Indeed, since the equation

$$\|h\|^2 = \|g\|^2 + \|f\|^2$$

implies that

$$\|g\| \leqq \|h\|, \tag{1}$$

we have

$$\|P\| \leqq 1.$$

But if $h \in G$, then $g = h$, so that there can be equality in (1). Therefore,

$$\|P\| = 1.$$

31. Properties of Projection Operators

From the definition of a projection operator it follows easily that

$$1)\ P^2 = P,$$

$$2)\ P^* = P.$$

Indeed, if $P = P_G$ then for an arbitrary $h \in H$ the vector $g = Ph$ belongs to G, so that $Pg = g$ and $P^2h = Ph$, and this implies that $P^2 = P$.

In order to prove that P is self-adjoint, we choose two arbitrary vectors $h_1, h_2 \in \mathrm{H}$ and let

$$h_1 = g_1 + f_1, \; h_2 = g_2 + f_2,$$

where $g_1 = Ph_1$ and $g_2 = Ph_2$. Then

$$(g_1, h_2) = (g_1, g_2) = (h_1, g_2),$$

so that

$$(Ph_1, h_2) = (h_1, Ph_2)$$

for $h_1, h_2 \in \mathrm{H}$. This implies that $P^* = P$.

From the properties just established it follows that

$$(Ph, h) \geqq 0.$$

In fact,

$$(Ph, h) = (P^2h, h) = (Ph, P^*h) = (Ph, Ph) \geqq 0.$$

Now we prove that the properties 1), 2) characterize a projection operator.

THEOREM: *If P is any operator defined on* H *such that, for arbitrary* $h_1, h_2 \in \mathrm{H}$,

(1) $$(P^2h_1, h_2) = (Ph_1, h_2),$$

(2) $$(Ph_1, h_2) = (h_1, Ph_2)$$

then there exists a subspace $\mathrm{G} \subset \mathrm{H}$ *such that P is the operator of projection on* G.

Proof: The operator P is bounded. This follows from (2) and a theorem in Section 25. However, it can be proved also by the following simple argument. We have

$$\| Ph \|^2 = (Ph, Ph) = (P^2h, h) = (Ph, h)$$

and

$$\| Ph \|^2 \leqq \| Ph \| \cdot \| h \|,$$

so that

$$\| Ph \| \leqq \| h \|.$$

Thus, the operator P is bounded and its norm is not greater than 1. We denote by G the set of all vectors $g \in \mathrm{H}$ for which

$$Pg = g.$$

Clearly, G is a linear manifold. We shall prove that G is closed so that it is also a subspace. Let $g_n \in \mathrm{G}$ $(n = 1, 2, 3, \ldots)$ and $g_n \to g$. Then

$$g_n = Pg_n$$

and

$$Pg - g_n = Pg - Pg_n = P(g - g_n),$$

so that

$$\| Pg - g_n \| \leqq \| g - g_n \|.$$

Let $n \to \infty$ to get

$$\|Pg - g\| \leqq 0,$$

so that

$$Pg = g.$$

Hence, $g \in G$, which implies that G is closed. We must prove that $P = P_G$, where P_G is the operator of projection on G. For each $h \in H$, the vector Ph belongs to G because $P(Ph) = Ph$. The subspace G also contains $P_G h$. Therefore, it is sufficient to prove that

$$(Ph - P_G h, g') = 0$$

or

$$(Ph, g') = (P_G h, g')$$

for each $g' \in G$. But this follows from the equations

$$(Ph, g') = (h, Pg') = (h, g'),$$
$$(P_G h, g') = (h, P_G g') = (h, g').$$

To conclude the present section, we remark that if G is a subspace and E is the identity operator, then $E - P$ is the operator of projection on $H \ominus G$.

32. Operations Involving Projection Operators

In the present section we shall prove a few simple propositions concerning the multiplication, addition and subtraction of projection operators.

THEOREM 1: *The product of two projection operators P_{G_1} and P_{G_2} is also a projection operator if and only if P_{G_1} and P_{G_2} commute, i.e., if*

$$P_{G_1} P_{G_2} = P_{G_2} P_{G_1}.$$

In this case,

$$P_{G_1} P_{G_2} = P_G,$$

where $G = G_1 \cap G_2$.[1]

Proof: First, let the product be a projection operator. Then

$$P_{G_1} P_{G_2} = (P_{G_1} P_{G_2})^* = P_{G_2}^* P_{G_1}^* = P_{G_2} P_{G_1}.$$

Fix $h \in H$ arbitrarily and let

$$g = P_{G_1} P_{G_2} h = P_{G_2} P_{G_1} h.$$

By the first representation $g \in G_1$ and, by the second, $g \in G_2$. Hence $g \in G_1 \cap G_2$. If $h \in G_1 \cap G_2$, then $P_{G_1} P_{G_2} h = h$. Thus, one half of the theorem is proved. Now assume that P_{G_1} and P_{G_2} commute. Let

$$P_{G_1} P_{G_2} = P_{G_2} P_{G_1} = P.$$

[1] A geometrical implication of the commutativity of the operators P_{G_1} and P_{G_2} is that the subspaces $G_1 \ominus (G_1 \cap G_2)$ and $G_2 \ominus (G_1 \cap G_2)$ are orthogonal.

It follows that

$$P^2 = (P_{G_1}P_{G_2})^2 = P_{G_1}P_{G_2}P_{G_1}P_{G_2} = P_{G_1}P_{G_1}P_{G_2}P_{G_2} = P_{G_1}P_{G_2} = P$$

and

$$(Ph_1, h_2) = (P_{G_1}P_{G_2}h_1, h_2) = (P_{G_2}h_1, P_{G_1}h_2)$$
$$= (h_1, P_{G_2}P_{G_1}h_2) = (h_1, P_{G_1}P_{G_2}h_2) = (h_1, Ph_2).$$

These equations show that the operator $P = P_{G_1}P_{G_2}$ satisfies the conditions of the theorem of the preceding section. Therefore, it is a projection operator.

COROLLARY: *Two subspaces* G_1 *and* G_2 *are orthogonal if and only if*

$$P_{G_1}P_{G_2} = 0.$$

THEOREM 2: *A finite sum of projection operators*

$$P_{G_1} + P_{G_2} + \ldots + P_{G_n} = Q \qquad (n < \infty)$$

is a projection operator if and only if

$$P_{G_i}P_{G_k} = 0 \qquad (i \neq k)$$

i.e., if and only if the spaces G_j $(j = 1, 2, 3, \ldots, n)$ *are pairwise orthogonal. In this case*

$$Q = P_G,$$

where

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_n.$$

Proof: If the spaces G_j are pairwise orthogonal, then $Q^2 = Q$, and, therefore, the sufficiency of the condition is evident. The last part of the assertion of the theorem is also evident. It remains only to prove the necessity of the condition. Let Q be a projection operator. Then

$$\|f\|^2 \geqq (Qf,f) = \sum_{j=1}^{n}(P_{G_j}f,f) \geqq (P_{G_i}f,f) + (P_{G_k}f,f)$$

for any pair of distinct indices i and k. From this relation it follows that

$$\|P_{G_i}f\|^2 + \|P_{G_k}f\|^2 \leqq \|f\|^2.$$

In this inequality let

$$f = P_{G_k}h.$$

Then

$$\|P_{G_i}P_{G_k}h\|^2 + \|P_{G_k}h\|^2 \leqq \|P_{G_k}h\|^2$$

which yields

$$\|P_{G_i}P_{G_k}h\| = 0$$

for $h \in H$. Thus,

$$P_{G_i} P_{G_k} = 0$$

so that the spaces G_i and G_k are orthogonal.

THEOREM 3: *The difference of two projection operators,*

(1) $$P_{G_1} - P_{G_2},$$

is a projection operator if and only if $G_2 \subset G_1$. *In this case* $P_{G_1} - P_{G_2}$ *is the operator of projection on* $G_1 \ominus G_2$.

Proof: In view of the remark at the end of the preceding section, we attempt to find conditions for which the difference

$$Q = E - (P_{G_1} - P_{G_2})$$

is a projection operator. Since the equation

$$Q = (E - P_{G_1}) + P_{G_2}$$

represents Q as the sum of two projection operators, it follows from Theorem 2 that

$$(E - P_{G_1}) P_{G_2} = 0$$

or, equivalently,

(2) $$P_{G_2} = P_{G_1} P_{G_2}.$$

If $g \in G_2$ then

$$g = P_{G_2} g = P_{G_1} P_{G_2} g = P_{G_1} g,$$

so that $g \in G_1$. Since every element $g \in G_2$ belongs to G_1, we have $G_2 \subset G_1$. This condition, which can be expressed in the form of (2), is necessary and sufficient in order that the difference (1) be a projection operator. It remains only to characterize the space G on which the operator (1) projects. The operator Q projects on

$$[H \ominus G_1] \oplus G_2.$$

Hence, the operator (1) projects on

(3) $$H \ominus \{[H \ominus G_1] \oplus G_2\},$$

i.e., on the subspace of vectors orthogonal both to G_2 and to $H \ominus G_1$. Since this subspace consists of all the vectors of G_1 which are orthogonal to G_2, it is the subspace

(4) $$G_1 \ominus G_2.$$

We notice that the difference (4) can be obtained directly from (3) by formally removing the brackets.

33. Monotone Sequences of Projection Operators

We shall prove that the relation $G_2 \subset G_1$ is equivalent to the inequality

$$\| P_{G_2} f \| \leqq \| P_{G_1} f \| \tag{1}$$

for all $f \in H$. The inequality (1) is evidently equivalent to

$$(P_{G_2} f, f) \leqq (P_{G_1} f, f)$$

or

$$(\{P_{G_2} - P_{G_1}\} f, f) \leqq 0$$

for $f \in H$. The last two inequalities are generally expressed by

$$P_{G_2} \leqq P_{G_1}.$$

Thus, we wish to prove that the relation $G_2 \subset G_1$ is equivalent to the relation $P_{G_2} \leqq P_{G_1}$. This will permit us to introduce for consideration monotone sequences of projection operators.

First, let $G_2 \subset G_1$. Then it follows that

$$P_{G_2} = P_{G_2} P_{G_1}.$$

Therefore, for each $f \in H$,

$$P_{G_2} f = P_{G_2} P_{G_1} f$$

and

$$\| P_{G_2} f \| \leqq \| P_{G_1} f \|. \tag{2}$$

Conversely, assume (2) for each $f \in H$. Consider

$$f = (E - P_{G_1}) h,$$

where h is an arbitrary element of H. From (2) and

$$P_{G_1}(E - P_{G_1}) h = 0,$$

we obtain

$$P_{G_2}(E - P_{G_1}) h = 0$$

or

$$P_{G_2} h = P_{G_2} P_{G_1} h.$$

Since this equality holds for each $h \in H$,

$$P_{G_2} = P_{G_2} P_{G_1},$$

so that $G_2 \subset G_1$. This completes the proof.

THEOREM: *If* P_{G_k} $(k = 1, 2, 3, \ldots)$ *is an infinite sequence of projection operators and if* $P_{G_k} \leqq P_{G_{k+1}}$ $(k = 1, 2, 3, \ldots)$, *then, as* $k \to \infty$, P_{G_k} *converges strongly to some projection operator* P.

Proof: For $m < n$ the difference $P_{G_n} - P_{G_m}$ is a projection operator. Therefore, for each $f \in H$,

$$\| P_{G_n} f - P_{G_m} f \|^2 = \| (P_{G_n} - P_{G_m}) f \|^2 = (\{P_{G_n} - P_{G_m}\} f, f) = \| P_{G_n} f \|^2 - \| P_{G_m} f \|^2. \tag{3}$$

Since, for fixed f, $\| P_{G_k} f \|^2$ increases with k but is bounded above by $\| f \|^2$, it has a finite limit. Hence, the right member of (3) tends to zero and the sequence $\{P_{G_n} f\}_{n=1}^{\infty}$ is fundamental in the sense of strong convergence. By the completeness of the space there exists the strong limit

$$f^* = \lim_{n \to \infty} P_{G_n} f.$$

We define the operator P by

$$f^* = Pf,$$

$f \in \mathrm{H}$. The operator P is obviously linear. Since

$$(P_{G_k} f, P_{G_k} g) = (P_{G_k} f, g) = (f, P_{G_k} g)$$

a passage to the limit yields

$$(Pf, Pg) = (Pf, g) = (f, Pg).$$

Therefore,

$$P = P^* = P^2,$$

so that P is a projection operator.

34. The Aperture of Two Linear Manifolds[2]

The present section is devoted to a concept which was introduced by B. Nagy and, independently of him, by M. G. Krein and M. A. Krasnoselski.[3]

DEFINITION: *The aperture of two linear manifolds in* H *is defined as the norm of the difference of the operators which project* H *on the closures of these two linear manifolds.*

The aperture of the linear manifolds M_1 and M_2 is denoted by the symbol $\Theta(\mathrm{M}_1, \mathrm{M}_2)$. Thus,

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) = \| P_1 - P_2 \| = \| P_2 - P_1 \|,$$

where P_1, P_2 are the operators of projection on the closed linear manifolds (subspaces) $\bar{\mathrm{M}}_1$, $\bar{\mathrm{M}}_2$, respectively. From the definition of aperture it follows that

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) = \Theta(\bar{\mathrm{M}}_1, \bar{\mathrm{M}}_2) = \Theta(\mathrm{H} \ominus \mathrm{M}_1, \mathrm{H} \ominus \mathrm{M}_2).$$

Consider the identity

$$P_2 - P_1 = P_2(E - P_1) - (E - P_2) P_1.$$

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[2] The results of this paragraph are necessary only for the construction of the theory of symmetric extensions (Chapter 7).

[3] M. G. Krein and M. A. Krasnoselski [1], B. Sz. Nagy [2].

It follows that, for $h \in H$,

$$(P_2 - P_1)h = P_2(E - P_1)h - (E - P_2)P_1h.$$

Since the vectors $P_2(E - P_1)h$ and $(E - P_2)P_1h$ are orthogonal,

$$(1) \quad \|(P_2 - P_1)h\|^2 = \|P_2(E - P_1)h\|^2 + \|(E - P_2)P_1h\|^2 \leqq$$
$$\leqq \|(E - P_1)h\|^2 + \|P_1h\|^2 = \|h\|^2.$$

This inequality shows that the aperture of two linear manifolds does not exceed 1:

$$\Theta(M_1, M_2) \leqq 1.$$

In addition to this, we see that the aperture is certainly equal to 1 if one of these manifolds contains a nonzero vector which is orthogonal to the other manifold. This fact implies the following criterion for the equality of the dimensions of two linear manifolds.

THEOREM: *If the aperture of the linear manifolds* M_1 *and* M_2 *is less than* 1, *then*

$$\dim M_1 = \dim M_2,$$

i.e., the two linear manifolds have the same dimension.

Proof: It is sufficient to prove that the inequality

$$\dim M_2 > \dim M_1$$

implies the existence of a nonzero vector in $\bar{M}_2$ which is orthogonal to the manifold $\bar{M}_1$. With this purpose in mind, we project $\bar{M}_1$ on $\bar{M}_2$. We get the subspace

$$G = P_2\bar{M}_1,$$

the dimension of which evidently does not exceed the dimension of the subspace $\bar{M}_1$ and, consequently, is less than the dimension of the subspace $\bar{M}_2$. Therefore, $\bar{M}_2 \ominus G$ contains a nonzero vector, i.e., $\bar{M}_2$ contains a nonzero vector which is orthogonal to G. This vector is orthogonal to the whole subspace $\bar{M}_1$ because the subspace $\bar{M}_1 \ominus G$ is orthogonal to $\bar{M}_2$.

The aperture of two linear manifolds has the equivalent definition,

$$(2) \quad \Theta(M_1, M_2) = \max\{\sup_{f \in \bar{M}_1, \|f\| = 1} \|(E - P_1)f\|, \sup_{g \in \bar{M}_1, \|g\| = 1} \|(E - P_2)g\|\}.$$

The quantity

$$\|(E - P_1)f\| = D[f, \bar{M}_1]$$

represents the distance between the point f and the manifold $\bar{M}_1$. The significance of this formula consists in the fact that it can be used to define the aperture of two linear manifolds not only in a Hilbert Space, but in any Banach space.[4]

[4] M. G. Krein, M. A. Krasnoselski, and D. P. Milman [1].

We now proceed with the proof of formula (2). According to the original definition of aperture and formula (1),

$$(3)\qquad \Theta(\mathrm{M}_1, \mathrm{M}_2) = \sup_{h\in\mathrm{H}} \frac{\|(P_2 - P_1)h\|}{\|h\|} =$$

$$= \sup_{h\in\mathrm{H}} \frac{\sqrt{\|P_2(E-P_1)h\|^2 + \|(E-P_2)P_1h\|^2}}{\|h\|}.$$

Consider the equation obtained from (3) by restricting the vector h to the subspace $\bar{\mathrm{M}}_1$. Then the right member is either unchanged or decreased, so that

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) \geqq \sup_{h\in\bar{\mathrm{M}}_1} \frac{\sqrt{\|P_2(E-P_1)h\|^2 + \|(E-P_2)P_1h\|^2}}{\|h\|} =$$

$$= \sup_{h\in\bar{\mathrm{M}}_1} \frac{\|(E-P_2)h\|}{\|h\|} = \rho_2.$$

In the same way it is proved that

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) \geqq \sup_{h\in\bar{\mathrm{M}}_2} \frac{\|(E-P_1)h\|}{\|h\|} = \rho_1.$$

Thus,

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) \geqq \max\{\rho_1, \rho_2\}$$

and it remains to prove that

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) \leqq \max\{\rho_1, \rho_2\}.$$

With this purpose in mind we remark that, by the definition of ρ_2,

$$(4)\qquad \|(E-P_2)P_1h\|^2 \leqq \rho_2^2 \|P_1h\|^2.$$

On the other hand

$$\begin{aligned}\|P_2(E-P_1)h\|^2 &= (P_2\{E-P_1\}h, P_2\{E-P_1\}h) = \\ &= (P_2\{E-P_1\}h, \{E-P_1\}h) = \\ &= (P_2\{E-P_1\}h, \{E-P_1\}^2h) = \\ &= (\{E-P_1\}P_2\{E-P_1\}h, \{E-P_1\}h) \leqq \\ &\leqq \|(E-P_1)P_2(E-P_1)h\| \cdot \|(E-P_1)h\|,\end{aligned}$$

and, consequently, according to the definition of ρ_1,

$$\|P_2(E-P_1)h\|^2 \leqq \rho_1 \|P_2(E-P_1)h\| \cdot \|(E-P_1)h\|,$$

so that

$$(5)\qquad \|P_2(E-P_1)h\| \leqq \rho_1 \|(E-P_1)h\|.$$

The inequalities (4) and (5) imply that

$$\begin{aligned}\|(E-P_2)P_1h\|^2 + \|P_2(E-P_1)h\|^2 &\leqq \rho_2^2\|P_1h\|^2 + \rho_1^2\|(E-P_1)h\|^2 \leqq \\ &\leqq \max\{\rho_1^2, \rho_2^2\}[\|P_1h\|^2 + \|(E-P_1)h\|^2] = \\ &= \|h\|^2 \max\{\rho_1^2, \rho_2^2\},\end{aligned}$$

so that formula (3) yields

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) \leqq \max\{\rho_1, \rho_2\}.$$

35. Unitary Operators

In three-dimensional Euclidean space the simplest operation after that of projection is rotation of the space, which changes neither the lengths of vectors nor the angles between pairs of them. We now consider an analogous operation in Hilbert space.

DEFINITION: *The operator* U *with domain* H ($\mathrm{D}_U = \mathrm{H}$) *and range* H ($\Delta_U = \mathrm{H}$) *is unitary if*

$$(Uf, Ug) = (f, g) \tag{1}$$

for $f, g \in \mathrm{H}$.

We emphasize that the given definition does not require that the operator be linear.

We prove first that the unitary operator has an inverse operator, which is also unitary. Recall that an operator T has an inverse if and only if $Tf = Tg$ implies that $f = g$ (cf. Section 14). Therefore, assume that $Uf = Ug$. Then

$$\begin{aligned} 0 &= (Uf - Ug, Uf - Ug) = \\ &= (Uf, Uf) - (Uf, Ug) - (Ug, Uf) + (Ug, Ug) = \\ &= (f,f) - (f,g) - (g,f) + (g,g) = (f - g, f - g), \end{aligned}$$

so that $f = g$. Thus, the operator U^{-1} exists. Since $\mathrm{D}_{U^{-1}} = \Delta_U$ and $\Delta_{U^{-1}} = \mathrm{D}_U$ the operator U^{-1} is defined in the whole space and maps it onto the whole space. Choose $f', g' \in \mathrm{H}$ arbitrarily and let $f = U^{-1}f'$, $g = U^{-1}g'$. Then

$$Uf = f', \; Ug = g',$$

and the substitution of these equations in (1) yields

$$(f', g') = (U^{-1}f', U^{-1}g')$$

so that U^{-1} is unitary.

From the proof just given it follows that, for $f, g \in \mathrm{H}$,

$$(Uf, g) = (f, U^{-1}g). \tag{2}$$

Indeed, let $U^{-1}g = g'$, so that $g = Ug'$. Then, since U is unitary,

$$(Uf, Ug') = (f, g'),$$

which is equivalent to (2).

We prove now that a unitary operator is necessarily linear. Let

$$f = \alpha_1 f_1 + \alpha_2 f_2.$$

Then, from (2),

$$\begin{aligned} (Uf, g) &= (f, U^{-1}g) = \alpha_1(f_1, U^{-1}g) + \alpha_2(f_2, U^{-1}g) = \\ &= \alpha_1(Uf_1, g) + \alpha_2(Uf_2, g) = (\alpha_1 Uf_1 + \alpha_2 Uf_2, g). \end{aligned}$$

Since g is arbitrary,

$$Uf = \alpha_1 Uf_1 + \alpha_2 Uf_2.$$

Therefore, every unitary operator is linear. Equation (2) shows that for a unitary operator, the adjoint operator coincides with its inverse:

$$U^* = U^{-1}.$$

Frequently the following simple proposition is useful: if a linear operator T satisfies the condition

$$(Tf, Tf) = (f, f) \tag{3}$$

and if $D_T = \Delta_T = H$, then T is unitary.

Proof: By condition (3),

$$(T\{f + \alpha g\}, T\{f + \alpha g\}) = (f + \alpha g, f + \alpha g).$$

Since T is linear,

$$(Tf, Tf) + \alpha(Tg, Tf) + \bar{\alpha}(Tf, Tg) + |\alpha|^2(Tg, Tg) = \\ = (f, f) + \alpha(g, f) + \bar{\alpha}(f, g) + |\alpha|^2(g, g).$$

Therefore, again by (3),

$$\alpha(Tg, Tf) + \bar{\alpha}(Tf, Tg) = \alpha(g, f) + \bar{\alpha}(f, g).$$

Since α is arbitrary,

$$(Tf, Tg) = (f, g),$$

so that T is a unitary operator.

36. Isometric Operators

Let H_1 and H_2 be two Hilbert spaces. We shall indicate the scalar product in H_1 by the index 1, and the scalar product in H_2 by the index 2.

DEFINITION: *The operator V with domain H_1 ($D_V = H_1$) and range H_2 ($\Delta_V = H_2$) is isometric if*

$$(Vf, Vg)_2 = (f, g)_1 \tag{1}$$

for $f, g \in H_1$.

In particular, H_1 and H_2 can be subspaces of a space H. In this case the indices on the scalar products are superfluous. Often the term isometric operator applies only in this special case, and the phrase isometric mapping is used in the general case.

A unitary operator in H is a special case of an isometric operator for which $H_1 = H_2 = H$. Many properties of unitary operators carry over to arbitrary isometric operators. We list now some of these properties, omitting those proofs which do not differ essentially from the proofs of corresponding properties for unitary operators.

1° Each isometric operator has an inverse operator which is also isometric.

2° If the operator V is linear, and maps all the space H_1 onto the space H_2 and if

$$(Vf, Vf)_2 = (f,f)_1$$

for $f \in H_1$, then V is an isometric operator.

3° Every isometric operator is linear.

Indeed, let $f', f'' \in H_1$, and $f = \alpha' f' + \alpha'' f''$. Then, for each $g \in H_1$,

$$(Vf, Vg)_2 = (f,g)_1 = \alpha'(f',g)_1 + \alpha''(f'',g)_1 = \\ = \alpha'(Vf', Vg)_2 + \alpha''(Vf'', Vg)_2 = (\alpha' Vf' + \alpha'' Vf'', Vg)_2.$$

Since $\Delta_V = H_2$, it follows that

$$Vf = \alpha' Vf' + \alpha'' Vf'';$$

hence, the operator V is linear.

Isometric operators were introduced implicitly in Section 9, where we introduced the concept of isomorphism of two Hilbert spaces.

To complete the present paragraph, we introduce an important concept which will be used repeatedly in what follows.

DEFINITION: *Let T_1 and T_2 be linear operators defined, respectively, in spaces H_1 and H_2, so that $D_{T_1} \subset H_1$, $\Delta_{T_1} \subset H_1$, $D_{T_2} \subset H_2$, $\Delta_{T_2} \subset H_2$. (In particular, the spaces H_1, H_2 may coincide). The operators T_1 and T_2 are called isomorphic or unitarily equivalent if there exists an isometric operator V, which maps H_1 onto H_2 and D_{T_1} onto D_{T_2}, such that $VT_1 f = T_2 Vf$ for each $f \in D_{T_1}$. In other words, T_1 and T_2 are unitarily equivalent if*

$$D_{T_2} = V D_{T_1}$$

and

$$T_1 = V^{-1} T_2 V.$$

37. The Fourier-Plancherel Operator

The object of this section is the proof of the Plancherel theorem, which may be stated as follows. *Let $g(t)$ be any function in $L^2(-\infty, \infty)$ and let*

$$(1) \qquad \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{e^{-ist} - 1}{-is} g(s)\, ds = h(t).$$

This formula defines a function $h(t)$ for almost all real t. Furthermore, $h(t) \in L^2(-\infty, \infty)$. The operator $\mathfrak{F}$ defined by $\mathfrak{F}g = h$, where h is given in terms of g by (1), *is unitary. Its inverse operator, $\mathfrak{F}^{-1}$, can be expressed by the formula*

$$g(t) = \mathfrak{F}^{-1} h(t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{e^{ist} - 1}{is} h(s)\, ds.$$

The operator $\mathfrak{F}$ is called the *Fourier-Plancherel operator*. If it is assumed that the function $g(t)$ is absolutely integrable on the entire real axis, then formula (1) can be replaced by

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} g(s)\, ds. \tag{2}$$

Thus, in this case $h(t)$ is a *Fourier integral* in the elementary sense.

The set of absolutely integrable functions in $L^2(-\infty, \infty)$ is mapped by the Fourier-Plancherel operator, expressed for this case either by (1) or (2), onto a linear manifold L which is dense in $L^2(-\infty, \infty)$. The value of the Plancherel theorem consists in the fact that it gives an extension of the elementary Fourier integral operator to an operator defined on all of $L^2(-\infty, \infty)$.

This approach to the Fourier-Plancherel operator leads to an equivalent definition. Let $g(t)$ be an arbitrary function in $L^2(-\infty, \infty)$ and let

$$g_N(t) = \begin{cases} g(t) & (-N \leqq t \leqq N) \\ 0 & (|t| > N). \end{cases}$$

Since the function $g_N(t)$ is absolutely integrable,

$$\mathfrak{F} g_N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} g_N(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ist} g(s)\, ds.$$

Since the operator $\mathfrak{F}$ is bounded,

$$\lim_{N \to \infty} \| \mathfrak{F} g - \mathfrak{F} g_N \| = 0$$

or

$$h(t) = \mathfrak{F} g(t) = \text{l. i. m.}_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ist} g(s)\, ds. \tag{3}$$

Hence, this alternate definition of the Fourier-Plancherel operator follows from (1).

It is easily proved that, in turn, definition (1) follows from definition (3). Indeed, for any element $f(t) \in L^2(-\infty, \infty)$,

$$\lim_{N \to \infty} (\mathfrak{F} g_N, f) = (\mathfrak{F} g, f). \tag{4}$$

Substituting

$$f(t) = \begin{cases} 1 & (0 \leqq t \leqq \tau), \\ 0 & (t < 0 \text{ and } t > \tau) \end{cases}$$

in (4), we obtain

$$\lim_{N \to \infty} \int_0^{\tau} dt \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-ist} g(s)\, ds = \int_0^{\tau} \mathfrak{F} g(t)\, dt.$$

After integration, passing to the limit and replacing τ by t, the last equation yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ist}-1}{-is} g(s)\, ds = \int_{0}^{t} \mathfrak{F}g(t)\, dt.$$

This relation implies that, for almost all real t,

$$\mathfrak{F}g(t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{e^{-ist}-1}{-is} g(s)\, ds.$$

Thus, formula (3) implies formula (1).

There exist various proofs of the theorem of Plancherel. The two essential elements of each proof are that the operator $\mathfrak{F}_0$, which is defined by equation (2) on the set $L \subset L^2(-\infty, \infty)$, does not change the norms of functions, and that it maps the set L onto a set which is dense in $L^2(-\infty, \infty)$. Extending the operator $\mathfrak{F}_0$ by continuity onto the whole space $L^2(-\infty, \infty)$, we get the operator $\mathfrak{F}$ which is defined by formula (3). Thus, the domain of definition of this operator is the whole space $L^2(-\infty, \infty)$. Since the operator $\mathfrak{F}$ does not change the norms of functions and since its range is dense in L^2, its range must also be the whole space.[5] To complete the proof it is sufficient to verify that the inverse operator results from the replacement of i by $-i$ in formula (1).

From the standpoint of the geometry of Hilbert space, it is particularly instructive to consider the set L_0 of functions

$$f(t) = P(t)e^{-\frac{t^2}{2}} \tag{5}$$

where $P(t)$ runs through the collection of all polynomials. The set L_0 is dense in $L^2(-\infty, \infty)$. Each function (5) can be represented in the form

$$f(t) = a_0 \varphi_0(t) + a_1 \varphi_1(t) + \ldots + a_n \varphi_n(t) \tag{6}$$

where the $\varphi_k(t)$ $(k = 0, 1, 2, \ldots)$ are the Tchebysheff-Hermite functions. By the orthogonality of these functions (cf. Section 11)

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \sqrt{\pi} \sum_{k=0}^{n} 2^k k! \, |a_k|^2.$$

Now we apply the operator $\mathfrak{F}_0$ to the function $f(t)$. For this purpose we make use of the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}(-1)^k e^{\frac{s^2}{2}} \frac{d^k}{ds^k} e^{-s^2} ds = i^k e^{\frac{t^2}{2}} \frac{d^k}{dt^k} e^{-t^2} \tag{7}$$

[5] Assuming the contrary, we could extend the inverse operator $\mathfrak{F}^{-1}$ onto the whole space $L^2(-\infty, \infty)$. The isometric operator thus obtained would take the identical values at at least two distinct points, which is impossible.

$$(k = 0, 1, 2, \ldots),$$

which will be proved later. The relation (7) implies that

$$\mathfrak{F}_0 \varphi_k = (-i)^k \varphi_k \qquad (k = 0, 1, 2, \ldots).$$

Therefore,

$$(6') \quad h(t) \equiv \mathfrak{F}_0 f(t) = a_0\varphi_0(t) + (-i)^1 a_1\varphi_1(t) + \ldots + (-i)^n a_n \varphi_n(t)$$

which yields

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

Thus, the operator $\mathfrak{F}_0$ does not change the norms of functions in the set L_0. Moreover, it follows from our considerations that the range of $\mathfrak{F}_0$ contains L_0, which is dense in $L^2(-\infty, \infty)$. Comparison of (6) and (6') proves also that in order to change $\mathfrak{F}_0$ to $\mathfrak{F}^{-1}$ it is necessary only to change i to $-i$.

It remains to prove relation (7). Here is the proof:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}(-1)^k e^{\frac{s^2}{2}} \frac{d^k}{ds^k} e^{-s^2} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2} \frac{d^k}{ds^k} \left(e^{-ist + \frac{s^2}{2}}\right) ds =$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-s^2} \frac{d^k}{ds^k} e^{\frac{(s-it)^2}{2}} ds =$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} i^k e^{-s^2} \frac{d^k}{dt^k} e^{\frac{(s-it)^2}{2}} ds =$$

$$= \frac{i^k}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \frac{d^k}{dt^k} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2} - \frac{t^2}{2} - ist} ds = i^k e^{\frac{t^2}{2}} \frac{d^k}{dt^k} e^{-t^2}.$$

Examples of operators defined on the space $L^2(0, \infty)$ are $\mathfrak{F}_c$ and $\mathfrak{F}_s$ where

$$\mathfrak{F}_c g(t) = \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^{\infty} \frac{\sin st}{s} g(s) ds = \sqrt{\frac{2}{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N g(s) \cos st \, ds,$$

$$\mathfrak{F}_s g(t) = \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^{\infty} \frac{1 - \cos st}{s} g(s) ds = \sqrt{\frac{2}{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N g(s) \sin st \, ds.$$

As an exercise we recommend that the reader verify that $\mathfrak{F}_c$ and $\mathfrak{F}_s$ are unitary. Since these operators satisfy the relations

$$\mathfrak{F}_c^* = \mathfrak{F}_c^{-1} = \mathfrak{F}_c,$$
$$\mathfrak{F}_s^* = \mathfrak{F}_s^{-1} = \mathfrak{F}_s,$$

they are self-adjoint.

Chapter IV

GENERAL CONCEPTS AND PROPOSITIONS IN THE THEORY OF LINEAR OPERATORS

38. Closed Operators

In chapter II, a general definition of a linear operator was given, but in the subsequent presentation we considered only bounded, i.e., continuous, operators defined everywhere in H. In the present chapter we begin a study of linear operators which are not necessarily continuous. The related but less restrictive requirement that given operators be *closed* is quite sufficient for many of our purposes.

DEFINITION: *An operator T (not necessarily linear) is closed if the relations*

$$f_n \in D_T, \ \lim_{n\to\infty} f_n = f, \ \lim_{n\to\infty} Tf_n = g$$

imply that

$$f \in D_T, \qquad Tf = g.$$

Thus, the difference between closedness and continuity consists of the following: if the operator T is continuous, then the existence of $\lim_{n\to\infty} f_n$ ($f_n \in D_T$) implies the existence of $\lim_{n\to\infty} Tf_n$; but if the operator T is only closed, then the convergence of the sequence

$$(1) \qquad f_1, f_2, f_3, \ldots \qquad (f_n \in D_T)$$

does not imply the convergence of the sequence

$$(2) \qquad Tf_1, Tf_2, Tf_3, \ldots .$$

However, if T is closed then, in particular, it has the property that two sequences of the type (2) cannot converge to different limits if the corresponding sequences (1) converge to the same limit.

An operator T having the property mentioned in the preceding sentence may not be closed; but it has *closed extensions*. Among these is the so-called *minimal closed extension*, which is contained in every closed extension of the operator T. The minimal closed extension is uniquely defined for each operator T. It is denoted by $\bar{T}$ and is called the *closure* of T.[1] In

[1] Translator's Note: The reader should verify that the operator $\bar{T}$ defined in the next sentence is closed. To do this, first prove that $\bar{\bar{T}} = \bar{T}$. The minimality of $\bar{T}$ is then easily established.

order to obtain $\bar{T}$, it is sufficient to adjoin to D_T all those elements $f \in \bar{D}_T$ which are limits of sequences (1) generating convergent sequences (2), and to require that

$$\bar{T}f = \lim_{N\to\infty} Tf_n.$$

It is not difficult to show (we leave this to the reader) the truth of the following assertion: if the operator T is closed, then each operator $T - \lambda E$ is closed, and if the inverse operator T^{-1} exists then it is closed.

39. The General Definition of an Adjoint Operator

In Section 22, in the definition of the adjoint operator for a given bounded operator A defined everywhere in H, we started from the fact that each element $g \in H$ uniquely determines an element g^* such that

$$(Af, g) = (f, g^*)$$

for all $f \in H$. Letting T denote an arbitrary operator, we consider once again the scalar product

(1) $$(Tf, g)$$

where f runs through D_T. We can no longer assert that for every element g the expression (1), as a function of the vector $f \in D_T$, is representable in the form

$$(f, g^*).$$

However, in general, there exist some pairs g and g^* for which

(2) $$(Tf, g) = (f, g^*)$$

for $f \in D_T$. Indeed, this equation holds at least for $g = g^* = 0$.

The existence of vectors g and g^*, for which (2) holds for each $f \in D_T$ is not sufficient to enable us to define an operator T^* which is adjoint to T. It is also necessary that the element g^* be determined uniquely by the element g. This last requirement is fulfilled if and only if D_T is dense in H. Indeed, if D_T is not dense in H then there is a nonzero element h which is orthogonal to D_T; then equation (2) implies that

$$(Tf, g) = (f, g^* + h)$$

for $f \in D_T$. On the other hand, if D_T is dense in H and if, for each $f \in D_T$,

$$(Tf, g) = (f, g_1^*),$$
$$(Tf, g) = (f, g_2^*),$$

then, for each $f \in D_T$,

$$(f, g_1^* - g_2^*) = 0.$$

This implies that $g_1^* = g_2^*$.

Thus, if D_T is dense in H, then the operator T has an adjoint operator T^*. Its domain D_{T^*} is defined as follows: $g \in D_{T^*}$ if and only if there exists a vector g^* such that (2) is satisfied for $f \in D_T$. For each such pair g and g^*,

$$T^*g = g^*.$$

We now list several simple propositions concerning adjoint operators, the proofs of which follow immediately from the definition.

1° The operator T^* is linear.

2° If $S \subset T$, then $S^* \supset T^*$.

3° The operator T^* is closed whether or not T is closed.

4° If the operator T has a closure $\bar{T}$, then $(\bar{T})^* = T^*$.

5° If the operator T^{**} exists, then

$$T \subset T^{**}.$$

The last proposition shows that a necessary condition for the existence of the operator T^{**} is the possibility of closing the operator T. The question as to whether this condition is sufficient for the existence of T^{**} we leave open until Section 46. There we shall consider the problem of the possibility of generalizing to the case of arbitrary operators the equation $T^{**} = T$, which was proved in Section 22 for bounded operators defined everywhere in H.

To complete the present paragraph, we consider the case of a linear operator T which has an inverse operator T^{-1}. Assume that D_T and $D_{T^{-1}}$ are dense in H. Then the operators T^* and $(T^{-1})^*$ exist. We shall prove that

$$(T^*)^{-1} = (T^{-1})^*.$$

To begin with let us assume that f runs through D_T and g runs through $D_{(T^{-1})^*}$. Then

$$(f, g) = (T^{-1}Tf, g) = (Tf, (T^{-1})^*g).$$

But this equation shows that

$$(T^{-1})^*g \in D_{T^*}$$

and

$$T^*(T^{-1})^*g = g. \tag{3}$$

On the other hand, if f runs through $D_{T^{-1}}$ and h runs through D_{T^*}, then

$$(f, h) = (TT^{-1}f, h) = (T^{-1}f, T^*h).$$

It follows that

$$T^*h \in D_{(T^{-1})^*}$$

and

$$(T^{-1})^*T^*h = h. \tag{4}$$

Relations (3) and (4) imply that

$$(T^*)^{-1} = (T^{-1})^*.$$

40. Eigenvectors, Invariant Subspaces and Reducibility of Linear Operators

A complex number λ is called an *eigenvalue* of the linear operator T if there exists a vector $f \neq 0$ such that

$$Tf = \lambda f. \tag{1}$$

Each such vector f is called an *eigenvector* of the operator T (more precisely, f is an eigenvector which belongs to the eigenvalue λ).

For each fixed eigenvalue λ of T the set consisting of all vectors which satisfy equation (1) is a linear manifold which contains at least one non-zero vector; each such set is called an *eigenmanifold* of T. Thus, the eigenmanifold corresponding to a given eigenvalue λ consists of the zero vector and all of the eigenvectors belonging to λ. The multiplicity of an eigenvalue λ is defined as the dimension (finite or infinite) of the corresponding eigenmanifold. If the operator T is closed then each eigenmanifold is closed, so that each eigenmanifold is a subspace.

More general than the concept of a closed eigenmanifold is the concept of an *invariant subspace*. A subspace $H_1 \subset H$ is called an invariant subspace of the operator T if every element of D_T belonging to H_1 is mapped by the operator T into an element also belonging to H_1, i.e., if the inclusion relation

$$f \in D_T \cap H_1$$

implies the inclusion relation

$$Tf \in H_1.$$

The operator T determines an operator T_1 defined in the subspace H_1 such that

$$D_{T_1} = D_T \cap H_1, \qquad T_1 \subset T.$$

This operator T_1 is called the *restriction* of the operator T to H_1.

If H_1 is an invariant subspace of the operator T, then its orthogonal complement $H \ominus H_1$ may or may not be an invariant subspace of this operator. For the moment, let us suppose that both H_1 and $H_2 = H \ominus H_1$ are invariant subspaces of the operator T, and let T_1 and T_2 be the restrictions of the operator T to H_1 and H_2, respectively. Does the study of the operator T reduce in this case to the study of the two operators T_1 and T_2? The answer is evidently in the affirmative if the operator T is defined everywhere in H. Indeed, for each element $h \in H$ we have a unique representation

$$h = h_1 + h_2$$

where $h_1 \in H_1$ and $h_2 \in H_2$, from which it follows that

$$Th = T_1 h_1 + T_2 h_2.$$

If the operator T is not defined everywhere in H, then the conclusion remains valid only under the additional condition that the projection P_{H_1} on H_1 does not map elements of D_T outside of D_T. Thus, we have the following proposition.

THEOREM 1: *If* H_1 *and its orthogonal complement* H_2 *are invariant subspaces of the operator* T, *and if* $P_{H_1}D_T \subset D_T$ *then, for each* $f \in D_T$,

$$Tf = T_1 f_1 + T_2 f_2,$$

where T_1 *and* T_2 *are the restrictions of* T *to* H_1 *and* H_2, *and* f_1 *and* f_2 *are the projections of* f *on* H_1 *and* H_2.

DEFINITION: *If the subspace* H_1 *satisfies the conditions of Theorem* 1, *then we say that it reduces the operator* T.

It is easy to see that if the subspace H_1 reduces the operator T, then its orthogonal complement H_2 also reduces T. Trivial subspaces which reduce T are the subspace $\{0\}$ and the space H itself. If the operator T does not have any other reducing subspaces, then it is said to be *irreducible*.

THEOREM 2: *Let* P *be the operator of projection on a given subspace* G. *Then* G *reduces* T *if and only if*

$$1) \quad Pf \in D_T \qquad \text{and} \qquad 2) \quad PTf = TPf$$

for $f \in D_T$, *i.e., if the operators* T *and* P *commute.*

Proof: We show first the necessity of the condition of the theorem. If the subspace G reduces T, then $f \in D_T$ implies that $Pf \in D_T$ so that 1) is proved. In order to establish 2), we suppose that

$$f = g + h,$$

where

$$g = Pf.$$

Since G reduces T,

$$Tf = Tg + Th$$

where $Tg \in G$ and $Th \in H \ominus G$. Therefore,

$$PTf = PTg = Tg$$

so that

$$PTf = TPf.$$

In much the same way, the sufficiency of the condition is easily proved.

We say that the projection operator P *reduces* T if the subspace G on which P projects reduces T.

The reduction of the study of the structure of the operator T to the investigation of its reducing subspaces and the restrictions of the operator T to them is based on the following proposition.

THEOREM 3: *Let the subspaces* H_k $(k = 1, 2, 3, \ldots, n;\ n \leqq \infty)$ *be pairwise orthogonal and let*

$$H = \sum_k \oplus H_k.$$

Let T *be a linear operator which is reduced by each of the subspaces* H_k. *Assume* T *closed if* $n = \infty$. *Finally, let* P_k *be the operator of projection on* H_k, *and* T_k *the restriction of* T *to* H_k. *Then* $f \in D_T$ *if and only if*

$$P_k f \in D_{Tk} \text{ and } \sum_{k=1}^{n} \| T_k P_k f \|^2 < \infty. \tag{2}$$

In this case

$$Tf = \sum_{k=1}^{n} T_k P_k f. \tag{3}$$

Proof: Let $f \in D_T$. Since H_k reduces T, we have $P_k f \in D_T$, which implies that

$$P_k f \in D_T \cap H_k = D_{Tk}.$$

In addition, $P_k Tf = TP_k f$. Therefore,

$$Tf = \sum_{k=1}^{n} P_k Tf = \sum_{k=1}^{n} TP_k f = \sum_{k=1}^{n} T_k P_k f.$$

In case $n = \infty$, this result implies the convergence of the series

$$\sum_{k=1}^{\infty} \| T_k P_k f \|^2.$$

Now let us assume conditions (2). If $n < \infty$, then the linearity of D_T implies $f \in D_T$ and equation (3). If $n = \infty$, then the linearity of D_T implies that each of the vectors

$$\sum_{k=1}^{r} P_k f \qquad (r = 1, 2, 3, \ldots)$$

belongs to D_T. Furthermore, the convergence of the series

$$\sum_{k=1}^{\infty} \| T_k P_k f \|^2$$

implies the convergence of the series

$$\sum_{k=1}^{\infty} T_k P_k f = \lim_{r \to \infty} T\left(\sum_{k=1}^{r} P_k f\right).$$

Since the operator T is closed,

$$f \in D_T \text{ and } Tf = \sum_{k=1}^{\infty} T_k P_k f.$$

Remark: The theorem remains valid in the more general case in which the space H is decomposed into an uncountable family of subspaces H_α. This follows from the fact that each of the vectors f and Tf has at most a countable set of nonzero projections on the subspaces H_α (cf. Section 9).

To complete the present paragraph, we consider a typical example. Let the space H be separable and let $\{e_k\}_{k=-\infty}^{\infty}$ be an orthonormal basis. We consider the linear operator U_0 which is defined for these elements by

$$U_0 e_k = e_{k+1} \qquad (-\infty < k < \infty)$$

and then extended to all of H by linearity and continuity; U_0 is a unitary operator. The closure G of the linear envelope of the set of vectors $\{e_k\}_{k=q}^{\infty}$, where q is fixed arbitrarily such that $q > -\infty$, is an invariant subspace of U_0. But G does not reduce U_0. Indeed, if P is the operator of projection on G, then

$$U_0 P e_{q-1} = 0, \qquad P U_0 e_{q-1} = P e_q = e_q,$$

so that

$$U_0 P \neq P U_0.$$

The operator U_0 is an example of an operator having no eigenvectors. Indeed if

$$U_0 f = \lambda f, \qquad f = \sum_{k=-\infty}^{\infty} \alpha_k e_k \neq 0$$

then

$$\sum_{k=-\infty}^{\infty} \alpha_k e_{k+1} = \lambda \sum_{k=-\infty}^{\infty} \alpha_k e_k$$

whence

$$\alpha_k = \lambda \alpha_{k+1} \qquad (-\infty < k < \infty).$$

Because

$$(f,f) = (U_0 f, U_0 f) = (\lambda f, \lambda f) = |\lambda|^2 (f,f)$$

and $f \neq 0$, we have $|\lambda| = 1$. Therefore,

$$|\alpha_k| = |\alpha_0| \qquad (\pm k = 1, 2, 3, \ldots),$$

which contradicts the fact that

$$0 < \sum_{k=-\infty}^{\infty} |\alpha_k|^2 < \infty.$$

Since U_0 has no eigenvectors, there does not exist a finite subspace reducing U_0. Later (cf. Section 48) we establish the existence of infinite subspaces which reduce U_0.

41. Symmetric Operators

A linear operator A is said to be symmetric[2] *if:*

(a) *its domain* D_A *is dense in* H; *and*

(b) *for* $f, g \in D_A$,

$$(Af, g) = (f, Ag).$$

From this definition it follows that the scalar product (Af, f) is real for $f \in D_A$. It may happen that

$$(Af, f) \geqq 0$$

for each $f \in D_A$. In this case a symmetric operator is said to be *positive.* A *negative* symmetric operator is defined analogously.

If a symmetric operator is bounded, then its extension by continuity is defined everywhere in H and is evidently a bounded self-adjoint operator (cf. Section 22). But if a symmetric operator is not bounded, then by the theorem in Section 25 its domain cannot be the whole space H.

If A is a symmetric operator, then evidently

$$A \subset A^*.$$

Since the adjoint operator is closed, this relation shows that a symmetric operator always has a closure.

If B is a symmetric extension of the operator A, then $B \subset A^*$, i.e., every symmetric extension of the operator A is a restriction of the adjoint operator A^*. Indeed, from $B \supset A$ it follows that $B^* \subset A^*$, and it remains merely to recall that $B \subset B^*$. An operator A which coincides with its adjoint ($A = A^*$) is said to be *self-adjoint*; it does not have a symmetric proper extension. A symmetric operator A not having symmetric proper extensions and not coinciding with its adjoint ($A \subsetneqq A^*$), is called a *maximal* symmetric operator.

THEOREM 1: *A symmetric operator A such that its range* Δ_A *is all of* H *is self-adjoint.*

Proof: It is sufficient to verify that every element $g \in D_{A^*}$ also belongs to D_A. Thus, let $g \in D_{A^*}$ and $A^*g = g^*$. Since $\Delta_A = H$, there exists an element $h \in D_A$ such that $Ah = g^*$. Consequently, for each $f \in D_A$,

$$(Af, g) = (f, g^*) = (f, Ah) = (Af, h).$$

Again since $\Delta_A = H$, we have $g = h$. So $g \in D_A$ and the theorem is proved.

COROLLARY: *If a bounded self-adjoint operator A has an inverse operator* A^{-1} *then* A^{-1} *is self-adjoint* (*bounded or unbounded*).

[2] The word "*Hermitian*" is often used in place of the word "symmetric."

Proof: It suffices to prove that the domain of A^{-1}, i.e., the range of A, is dense in H. Assume, therefore, that the range of A is not dense in H. Then there exists a vector $h \neq 0$ orthogonal to Δ_A, so that $(f, Ah) = (Af, h) = 0$, for every $f \in \mathrm{H}$. It follows that $Ah = 0$, which contradicts the existence of the inverse operator.

THEOREM 2: *The eigenvalues of a symmetric operator are real.*

Proof: If

$$Af = \lambda f \quad (f \neq 0),$$

then

$$\lambda(f,f) = (\lambda f, f) = (Af, f) = (f, Af) = (f, \lambda f) = \bar{\lambda}(f,f),$$

whence

$$\lambda = \bar{\lambda}.$$

THEOREM 3: *Eigenvectors f_1 and f_2 belonging to the two different eigenvalues λ_1 and λ_2 of a symmetric operator are orthogonal.*

Proof: Letting

$$Af_1 = \lambda f_1, \qquad Af_2 = \lambda_2 f_2$$

where $\lambda_1 \neq \lambda_2$, we obtain

$$\lambda_1(f_1, f_2) = (\lambda_1 f_1, f_2) = (Af_1, f_2) = (f_1, Af_2) = (f_1, \lambda_2 f_2) = \lambda_2(f_1, f_2).$$

Therefore

$$(\lambda_1 - \lambda_2)(f_1, f_2) = 0$$

and

$$(f_1, f_2) = 0.$$

THEOREM 4: *If* G *is an invariant subspace of the symmetric operator A and if the projection P on* G *satisfies the relation $P\mathrm{D}_A \subset \mathrm{D}_A$ then the subspace* G *reduces the operator A.*

Proof: In view of Theorem 1 of Section 40, the only thing to prove is that $\mathrm{H} \ominus \mathrm{G}$ is an invariant subspace of the operator A. According to the conditions of the present theorem, if $f \in \mathrm{D}_A \cap (\mathrm{H} \ominus \mathrm{G})$ and $g \in \mathrm{D}_A$, then

$$(Af, Pg) = 0$$

so that

$$(PAf, g) = 0.$$

Since D_A is dense in H, this implies that

$$PAf = 0$$

and

$$Af \in \mathrm{H} \ominus \mathrm{G},$$

so that the theorem is proved.

In conclusion we present the following lemma, which will be needed later (in Appendix II).

LEMMA: *In order that a linear manifold* D ($D_A \subset D \subset D_{A^*}$) *be the domain of a self-adjoint extension of a given symmetric operator* A, *it is necessary and sufficient that* D *be the set of elements* $f \in D_{A^*}$ *which satisfy the condition*

$$(A^*f, g) = (f, A^*g)$$

for each $g \in D$.

The proof of this lemma follows immediately from the considerations of the present section.

42. More about Isometric and Unitary Operators

In this section we consider isometric operators in the narrow sense, i.e., the domain D_V and the range Δ_V of the operator V are subspaces of the same space H. An isometric operator is said to be *maximal* if it does not have an isometric proper extension.

THEOREM 1:[3] *Each eigenvalue of an isometric operator* V *has absolute value one.*

Proof: Let

$$Vf = \lambda f,$$

where $f \neq 0$. Then

$$(f,f) = (Vf, Vf) = (\lambda f, \lambda f) = |\lambda|^2 (f,f),$$

whence $|\lambda| = 1$, because $(f,f) \neq 0$.

THEOREM 2: *Eigenvectors* f_1 *and* f_2 *belonging to two different eigenvalues* λ_1 *and* λ_2 *of an isometric operator* V *are orthogonal.*

Proof: Let

$$Vf_1 = \lambda_1 f_1, \qquad Vf_2 = \lambda_2 f_2,$$

where $\lambda_1 \neq \lambda_2$. Then

$$(f_1,f_2) = (Vf_1, Vf_2) = (\lambda_1 f_1, \lambda_2 f_2) = \lambda_1 \bar{\lambda}_2 (f_1,f_2)$$

so that

$$(1 - \lambda_1 \bar{\lambda}_2)(f_1,f_2) = 0.$$

Then

$$(f_1,f_2) = 0,$$

since $1 - \lambda_1 \bar{\lambda}_2 \neq 0$.

THEOREM 3: *In order that a subspace* G *reduce a unitary operator* U, *it is necessary and sufficient that* G *be an invariant subspace of each of the operators* U *and* U^{-1}.

Proof: Let G reduce U. Then $H \ominus G$ is an invariant subspace of the operator U and, for each $f \in H \ominus G$ and each $g \in G$,

$$(Uf, g) = 0,$$

[3] In essence this fact was established already in Section 40.

so that

$$(f, U^{-1}g) = 0.$$

This implies that $U^{-1}g \in G$, so that G is an invariant subspace of U^{-1}. Conversely, let G be invariant with respect to the operator U^{-1}. Then, for $f \in H \ominus G$ and $g \in G$,

$$(Uf, g) = (f, U^{-1}g) = 0.$$

It follows that the subspace $H \ominus G$ is invariant with respect to the operator U. By an argument similar to that in Theorem 4 of Section 41, G is invariant with respect to U. Consequently, G reduces the operator U.

43. The Concept of the Spectrum (Particularly of a Self-Adjoint Operator)

In linear algebra, the spectrum of a quadratic form is defined as the set of its eigenvalues. Analogously, in the elementary theory of integral equations the spectrum of an equation is defined as the set of eigenvalues of the equation, i.e., the set of eigenvalues of the corresponding integral operator. In this case it turns out that certain nonhomogeneous equations (vector or functional) containing a parameter λ are uniquely solvable for any right member if λ does not belong to the spectrum. These equations are not usually solvable if λ belongs to the spectrum.

We turn now to some general considerations. Suppose that we are given a closed linear operator T defined on a manifold D_T which is dense in H. Let λ denote a parameter which can assume any complex value and consider the operator equation,

$$Tf - \lambda f = g.$$

The study of this equation reduces to an investigation of the linear manifold $\Delta_T(\lambda)$ which consists of the vectors $(T - \lambda E)f$ where f runs through D_T. Thus, $\Delta_T(\lambda)$ is the range of the operator $T - \lambda E$. It can be expressed in the form $\Delta_T(\lambda) = (T - \lambda E)D_T$. The operator $T - \lambda E = T_\lambda$ defines a correspondence (not necessarily one-to-one) between D_T and $\Delta_T(\lambda)$. If this correspondence is one-to-one, then the operator $T - \lambda E$ has an inverse operator $(T - \lambda E)^{-1}$ with domain $\Delta_T(\lambda)$ and range D_T.

DEFINITION 1: *If $(T - \lambda E)^{-1}$ exists and is a bounded operator defined everywhere in* H ($\Delta_T(\lambda) = H$), *then λ is called a regular value (or regular point) of the operator T. All other points of the complex plane comprise the spectrum of the operator T.*

In each of the cases mentioned above, which are related to linear algebra and the elementary theory of integral equations, the spectrum of an operator consists of all of its eigenvalues. But, in more general situa-

tions, the collection of eigenvalues does not exhaust the spectrum. Indeed, Theorem 1 stated below characterizes the eigenvalues of an operator T as the complex numbers λ for which either the operator $T - \lambda E$ does not have an inverse, or $(T - \lambda E)^{-1}$ exists but is not a bounded operator defined on all of H.

THEOREM 1: *The correspondence between* D_T *and* $\Delta_T(\lambda)$ *determined by the operator* $T - \lambda E$ *is one-to-one if and only if* λ *is not an eigenvalue of the operator* T.

Proof: If the operator $T - \lambda E$ does not determine a one-to-one correspondence between D_T and $\Delta_T(\lambda)$, then there exist $f_1, f_2 \in D_T$ such that $f_1 \neq f_2$ and

$$Tf_1 - \lambda f_1 = g, \quad Tf_2 - \lambda f_2 = g.$$

Consequently,

$$Tf = \lambda f,$$

where $f = f_1 - f_2 \neq 0$, so that λ is an eigenvalue of the operator T. The proof of the converse assertion is also simple and is left to the reader.

Instead of considering the general case, which includes all possible hypotheses concerning the operator $(T - \lambda E)^{-1}$ and the domain $\Delta_T(\lambda)$, we restrict ourselves to the important special case in which the original operator is self-adjoint (we shall denote this operator not by T, but by A.)

THEOREM 2: *The number* λ *is an eigenvalue of the self-adjoint operator* A *if and only if*

$$\overline{\Delta_A(\lambda)} \neq \mathrm{H}.$$

Proof: Let λ be an eigenvalue of A, so that

$$Af = \lambda f \qquad (f \neq 0).$$

Then, for each $h \in D_A$,

$$(f, \{A - \lambda E\} h) = (Af - \lambda f, h) = 0,$$

which implies that

$$f \perp \Delta_A(\lambda).$$

But this is possible only when $\overline{\Delta_A(\lambda)} \neq \mathrm{H}$. We suppose now that $\overline{\Delta_A(\lambda)} \neq \mathrm{H}$. Then there exists a nonzero vector f which is orthogonal to the manifold $\Delta_A(\lambda)$. Therefore, for each $h \in D_A$,

$$(f, \{A - \lambda E\} h) = 0.$$

It follows that $f \in D_{A^*}$ and

$$A^* f = \bar{\lambda} f.$$

But $A^* = A$, so that

$$Af = \bar{\lambda} f,$$

i.e., $\bar{\lambda}$ is an eigenvalue of the operator A. Finally, $\bar{\lambda} = \lambda$ because the eigenvalues of a self-adjoint operator are real.

We note that in the course of the proof of Theorem 2, we have proved the following proposition.

THEOREM 2*: *The eigenmanifold of an operator A corresponding to the eigenvalue λ is the orthogonal complement of the linear manifold* $\Delta_A(\lambda) = (A - \lambda E)\mathrm{D}_A$.

THEOREM 3: *Nonreal points λ in the complex plane are regular points of a self-adjoint operator A.*

Proof: The number $\lambda = \xi + i\eta$ $(\eta \neq 0)$ cannot be an eigenvalue of the operator A. Therefore, on the basis of Theorem 1, the operator $(A - \lambda E)^{-1}$ exists. Letting

$$(A - \lambda E)f = g,$$

we obtain

$$\begin{aligned}\|g\|^2 &= (\{A - \xi E\}f - i\eta f, \{A - \xi E\}f - i\eta f) = \\ &= \|\{A - \xi E\}f\|^2 + i\eta(\{A - \xi E\}f, f) - i\eta(f, \{A - \xi E\}f) + \eta^2\|f\|^2 = \\ &= \|\{A - \xi E\}f\|^2 + \eta^2\|f\|^2,\end{aligned}$$

whence

$$\|f\| \leq \frac{1}{|\eta|}\|g\|,$$

i.e.,

$$\|(A - \lambda E)^{-1}g\| \leq \frac{1}{|\eta|}\|g\|.$$

Since this relation is valid for every $g \in \Delta_A(\lambda)$, the operator $(A - \lambda E)^{-1}$ is bounded. By Theorem 2 and the fact that λ is not an eigenvalue of the operator A,

$$\overline{\Delta_A(\lambda)} = \mathrm{H}.$$

It remains to show that the manifold $\Delta_A(\lambda)$ is closed. Supposing that $\Delta_A(\lambda) \neq \overline{\Delta_A(\lambda)}$ we extend the bounded operator $(A - \lambda E)^{-1}$ by continuity to $\overline{\Delta_A(\lambda)}$. This extension coincides with the closure of the operator $(A - \lambda E)^{-1}$, which therefore is not closed. But this is impossible because the closedness of the operator A implies the closedness of the operator $(A - \lambda E)^{-1}$.

COROLLARY 1: *The spectrum of a self-adjoint operator is a subset of the real axis.*

COROLLARY 2: *A regular point of the self-adjoint operator A can be defined as a value of the parameter λ for which* $\Delta_A(\lambda) = \mathrm{H}$.

Proof: If λ is nonreal then it is regular by Theorem 3. If λ is real and $\Delta_A(\lambda) = \mathrm{H}$ then, by Theorem 2, λ is not an eigenvalue of the operator A. Therefore, by Theorem 1, there exists the inverse operator $(A - \lambda E)^{-1}$ defined everywhere in H. This operator is self-adjoint and consequently

(cf. the beginning of Section 41) it is bounded, so that Definition 1 applies.

Now, without contradicting Definition 1, we can use the following definition.

DEFINITION 2: *If A is a self-adjoint operator, then the point λ is a regular point of A if $\Delta_A(\lambda) = \mathrm{H}$ and λ is a point of the spectrum if $\Delta_A(\lambda) \neq \mathrm{H}$.*

A further refinement of the concept of the spectrum of a self-adjoint operator is given by the following definition.

DEFINITION 3: *We say that the point λ belongs to the point (discrete) spectrum of the self-adjoint operator A if $\overline{\Delta_A(\lambda)} \neq \mathrm{H}$ and λ belongs to the continuous spectrum if $\Delta_A(\lambda) \neq \overline{\Delta_A(\lambda)}$.*

The possibility that λ belongs to the point spectrum and also to the continuous spectrum (for $\overline{\Delta_A(\lambda)} \neq \Delta_A(\lambda) \neq \mathrm{H}$) is not excluded.

On the basis of the theorem 2, the point spectrum of a self-adjoint operator coincides with the set of its eigenvalues.

To complete the present paragraph, we prove the following proposition.

THEOREM 4: *The spectrum of a self-adjoint operator is a closed set.*

Proof: It is sufficient to show that the set of regular points of a self-adjoint operator A is open. Let λ_0 be a regular point. Then there exists a number $k > 0$ such that

$$\| Af - \lambda_0 f \| \geqq k \| f \|$$

for $f \in \mathrm{D}_A$. If $0 < \delta \leqq \frac{k}{2}$ then, for $|\lambda - \lambda_0| \leqq \delta$ and $f \in \mathrm{D}_A$,

$$\| Af - \lambda f \| \geqq \| Af - \lambda_0 f \| - \delta \| f \| \geqq \frac{1}{2} k \| f \|.$$

Therefore, in the first place, λ is not an eigenvalue of the operator A, so that $\overline{\Delta_A(\lambda)} = \mathrm{H}$, and, in the second place, the inverse operator $(A - \lambda E)^{-1}$ is bounded. The equation $\Delta_A(\lambda) = \overline{\Delta_A(\lambda)}$ is a result of the fact that A is closed. Thus, every point λ such that $|\lambda - \lambda_0| \leqq \delta$ is regular, and the theorem is proved.

44. The Resolvent

As in the preceding Section, we begin with an arbitrary closed linear operator T, the domain of which is dense in H, and specialize later to a self-adjoint operator A. The operator $R_\lambda = (T - \lambda E)^{-1}$, which depends on the parameter λ, is called the *resolvent* of the operator T, and is defined for all the values λ for which it exists and for which its domain of definition, i.e., $\Delta_T(\lambda)$, is dense in H.

For each regular point of the operator T the resolvent R_λ is a bounded operator defined on the whole space. The operator R_λ determines a one-to-one correspondence between $\Delta_T(\lambda)$ and D_T. In particular, if λ is a regular point of the operator T then $R_\lambda h = 0$ if and only if $h = 0$.

THEOREM 1: *For any two regular points λ and μ of the operator T the equation,*

$$R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda,$$

is satisfied. (This is the so-called Hilbert relation.)

Proof: Since λ and μ are regular points of the operator T we have, for each $h \in H$,

$$R_\lambda h = R_\mu (T - \mu E) R_\lambda h,$$
$$R_\mu h = R_\mu (T - \lambda E) R_\lambda h.$$

Subtracting the first equation from the second, we get the desired relation.

From the Hilbert relation follows the commutativity of the resolvents corresponding to any pair of regular values μ, λ:

$$R_\lambda R_\mu = R_\mu R_\lambda.$$

This is a special case of the following general proposition.

THEOREM 2: *In order that the operator T commute with a given bounded operator S which is defined everywhere in* H, *it is necessary that S commute with the resolvent $R_\lambda = (T - \lambda E)^{-1}$ for each regular value λ, and it is sufficient that S and R_λ commute for at least one regular value λ.*

Proof: We suppose first that the operators T and S commute, i.e., that

$$TSf = STf$$

for $f \in D_T$. If λ is a regular point of the operator T and

$$f = R_\lambda h,$$

then f runs through D_T as h runs through H. But, the commutativity of the operators T and S implies that

$$(T - \lambda E) Sf = S(T - \lambda E) f$$

for each $f \in D_T$, which yields

$$R_\lambda (T - \lambda E) Sf = R_\lambda S(T - \lambda E) f.$$

Since $R_\lambda (T - \lambda E) = E$ and $(T - \lambda E) f = h$, we obtain

$$SR_\lambda h = R_\lambda Sh.$$

Since proof of the second part of the assertion is quite simple, it is omitted.

We turn now to an arbitrary self-adjoint operator A. We shall define its resolvent also for eigenvalues (i.e., for points of the discrete spectrum), after which the resolvent of A will be defined for all points of the λ-plane.

With this aim, we assume that λ' is an eigenvalue of the operator A

and we denote by $G_{\lambda'}$ the corresponding eigenmanifold. As we know, $G_{\lambda'}$ reduces the operator A. Let A' be the restriction of the operator A to $H \ominus G_{\lambda'} = H'$. It is easy to see that A' is a self-adjoint operator in H' for which λ' is not an eigenvalue. We define $R_{\lambda'}$ by

$$R_{\lambda'} = (A' - \lambda' E)^{-1}.$$

The domain of the operator $R_{\lambda'}$ is the manifold $\Delta_{A'}(\lambda')$, which is dense in H'. The manifold $\Delta_{A'}(\lambda')$ consists of all vectors which can be represented in the form

$$(A' - \lambda' E)f' = (A - \lambda' E)f',$$

with $f' \in D_{A'}$. But it is easy to see that this manifold also consists of all vectors of the form

$$(A - \lambda' E)f$$

for $f \in D_A$. Thus,

$$\Delta_{A'}(\lambda') = \Delta_A(\lambda').$$

The range of the operator $R_{\lambda'}$ is obtained if D_A is projected on the orthogonal complement of the eigenmanifold corresponding to λ'.

The manner in which the resolvent R_λ of a self-adjoint operator depends on the value of the parameter λ is shown in the following table:

λ is a regular point of the the operator.	R_λ is a bounded operator defined everywhere in H.
λ belongs to the point spectrum, but does not belong to the continuous spectrum.	R_λ is a bounded operator defined on a set which is not dense in H.
λ belongs to the continuous spectrum but does not belong to the point spectrum.	R_λ is an unbounded operator defined on a set which is dense in H.
λ belongs both to the point spectrum and to the continuous spectrum.	R_λ is an unbounded operator defined on a set which is not dense in H.

To complete the present section, we show that

$$(R_\lambda)^* = R_{\bar{\lambda}} \tag{1}$$

if and only if λ does not belong to the point spectrum of the operator (if it does, then the operator R_λ does not have an adjoint). If λ is real, then $R_\lambda = (A - \lambda E)^{-1}$ is a self-adjoint operator, and (1) is obvious. Suppose λ is not real. Then, for $f, g \in H$,

$$(R_\lambda f, g) = (R_\lambda f, \{A - \bar{\lambda}E\}R_{\bar{\lambda}}g) = (\{A - \lambda E\}R_\lambda f, R_{\bar{\lambda}}g) = (f, R_{\bar{\lambda}}g),$$

which implies formula (1).

45. Conjugation Operators[4]

A *conjugation operator* is an operator I defined on H such that

1) $(If, Ig) = \overline{(f, g)}$,

2) $I^2 f = f$

for $f, g \in \mathrm{H}$. From 2) it follows that the range of the operator I is the whole space H. In fact, each vector $h \in \mathrm{H}$ can be represented in the form $h = Ig$ merely by taking $g = Ih$. Instead of the usual linearity, the operator I has the following property, which is sometimes called conjugate linearity:

$$I(\alpha f + \beta g) = \bar{\alpha} If + \bar{\beta} Ig.$$

Indeed, letting

$$g = Ih$$

in 1), we get

$$(If, h) = \overline{(f, Ih)}.$$

It follows that

$$(I(\alpha f + \beta g), h) = \overline{(\alpha f + \beta g, Ih)} = \bar{\alpha}\overline{(f, Ih)} + \bar{\beta}\overline{(g, Ih)} =$$
$$= \bar{\alpha}(If, h) + \bar{\beta}(Ig, h) = (\bar{\alpha} If + \bar{\beta} Ig, h),$$

and the assertion is proved.

An example of a conjugation operator in L^2 is the operation of transition to the complex conjugate function:

$$I\varphi(t) = \overline{\varphi(t)}.$$

For each conjugation operator in a separable space it is possible to select an orthonormal basis $\{e_k\}_1^\infty$ such that if

$$f = \sum_{k=1}^{n} x_k e_k,$$

then

$$If = \sum_{k=1}^{\infty} \bar{x}_k e_k.$$

The proof of this simple fact we leave to the reader.

DEFINITION: *A symmetric operator A is said to be real with respect to a given conjugation operator I, if the operators A and I commute, i.e., if $f \in \mathrm{D}_A$ implies that $If \in \mathrm{D}_A$ and*

$$IAf = AIf.$$

THEOREM: *If R_λ is the resolvent of a self-adjoint operator A which is real with respect to a given conjugation operator I, then for each nonreal λ*

(1) $$R_\lambda = IR_\lambda^* I.$$

[4] The results of the present section will be used only in Appendix II.

Proof: We apply the conjugation operator to both members of the equation

$$(A - \lambda E)R_\lambda = E$$

to get

$$I(A - \lambda E)R_\lambda = I.$$

Since A is real with respect to I,

$$(A - \bar{\lambda} E)IR_\lambda = I.$$

Now we apply to both members the operator $IR_{\bar{\lambda}}$. This gives

$$R_\lambda = IR_{\bar{\lambda}}I.$$

Since $R_{\bar{\lambda}} = R_\lambda^*$, the proof is complete.

If T is a linear operator with domain dense in H and I is a conjugation operator, then, by analogy with the terminology of the theory of matrices, the operator $IT^*I = T'$ is called the transposition with T. Now (1) can be written in the form

$$R'_\lambda = R_\lambda.$$

46. The Graph of an Operator

We consider the set of *ordered pairs* $\{f, g\}$ where the *abscissa* f and the *ordinate* g run through the Hilbert space H. A vector space is defined on this set by means of the equations

$$\alpha\{f,g\} = \{\alpha f, \alpha g\}, \quad \{f_1,g_1\} + \{f_2,g_2\} = \{f_1 + f_2, g_1 + g_2\}$$

(the zero element is $\{0, 0\}$). With the scalar product defined by

$$(\{f_1,g_1\},\{f_2,g_2\}) = (f_1,f_2) + (g_1,g_2)$$

this vector space becomes a Hilbert space, which we denote by **H**.

Let T be an operator in H. Then the set $\mathbf{M}(T)$ of all points of the form $\{f, Tf\}$ is called the *graph* of the operator T. Every point of the set $\mathbf{M}(T)$ is determined uniquely by its abscissa. Conversely, if all the points of a set **M** in **H** are determined uniquely by their abscissae, then **M** is the graph of some operator in **H**.

Whether or not an operator is closed is reflected in a very simple manner by its graph. In fact, an operator T in H is closed if and only if its graph $\mathbf{M}(T)$ is a closed subset of **H**. Starting from the fact that every subset of **H** has a closure one might be tempted to infer that every operator T has a closure. The fault in this reasoning is that the closure $\overline{\mathbf{M}(T)}$ of the graph of T may contain points which are not determined uniquely by their abscissae. In this case, $\overline{\mathbf{M}(T)}$ is not the graph of any operator. However, if the set $\overline{\mathbf{M}}(T)$ does not contain two distinct points with identical abscissae, then the operator T has a closure $\bar{T}$ and $\mathbf{M}(\bar{T}) = \overline{\mathbf{M}(T)}$.

It is easily seen that if the operator T is linear, then the set $\mathbf{M}(T)$ is a linear manifold in H.

We now define an operator $\boldsymbol{U}$ on $\mathbf{H}$ by

$$\boldsymbol{U}\{f,g\} = \{ig, -if\}.$$

The operator $\boldsymbol{U}$ is unitary since its range is the whole space $\mathbf{H}$ and

$$(\boldsymbol{U}\{f_1,g_1\}, \boldsymbol{U}\{f_2,g_2\}) = (\{ig_1, -if_1\}, \{ig_2, -if_2\}) =$$
$$= (g_1,g_2) + (f_1,f_2) = (\{f_1,g_1\}, \{f_2,g_2\}).$$

We remark also that $\boldsymbol{U}^2 = \boldsymbol{E}$, where $\boldsymbol{E}$ is the identity operator in $\mathbf{H}$.

We now consider the questions raised in Section 39 concerning the existence of T^{**} and the validity of the equation $T^{**} = \bar{T}$. The graph of T will be used in the analysis of these questions. To begin with, we consider the equation

$$(Tf,g) - (f,g^*) = -i(\{iTf, -if\}, \{g,g^*\}) = -i(\boldsymbol{U}\{f,Tf\}, \{g,g^*\}),$$

which holds for each $f \in \mathrm{D}_T$ and any pair of elements g and g^* in H. This equation has the following consequences:

1° In order that the elements g, $g^* \in \mathrm{H}$ satisfy the equation,

$$(Tf,g) = (f,g^*),$$

for each $f \in \mathrm{D}_T$, it is necessary and sufficient that the element $\{g, g^*\}$ of the space $\mathbf{H}$ be orthogonal to the image $\boldsymbol{U}\mathbf{M}(T)$ under $\boldsymbol{U}$ of the graph of the operator T.

2° If the operator T^* exists, then its graph is given by

$$\mathbf{M}(T^*) = \mathbf{H} \ominus \overline{\boldsymbol{U}\mathbf{M}(T)}.$$

3° The operator T^* exists if and only if each point of the set

$$\mathbf{H} \ominus \overline{\boldsymbol{U}\mathbf{M}(T)}$$

is determined uniquely by its abscissa.

Consequence 3° is another criterion for the existence of the adjoint operator (the first criterion, the density of the manifold D_T in H, was established in Section 39).

THEOREM 1: *If the linear operator T with domain dense in* H *has a closure then the operator T^{**} exists and is the closure* [5] *of the operator T:*

$$T^{**} = \bar{T}.$$

(*In particular, if the operator T is closed and $\bar{\mathrm{D}}_T = \mathrm{H}$, then $T = T^{**}$.*)

Proof: We assume, at first, that the operator T is closed. Then the set $\mathbf{M}(T)$ is closed, so that $\boldsymbol{U}\mathbf{M}(T)$ also is closed. Therefore, the relation in consequence 2° can be written in the form

$$\mathbf{H} = \boldsymbol{U}\mathbf{M}(T) \oplus \mathbf{M}(T^*).$$

[5] Thus, Theorem 1 gives a method for finding the closure of an operator. This method is used sometimes in applications (cf. Section 49).

Hence, applying the operator $\boldsymbol{U}$, we get

$$\mathbf{H} = \mathbf{M}(T) \oplus \boldsymbol{U}\mathbf{M}(T^*) \tag{1}$$

which implies that

$$\mathbf{H} \ominus \boldsymbol{U}\mathbf{M}(T^*) = \mathbf{M}(T). \tag{2}$$

Since the points of the graph $\mathbf{M}(T)$ are determined uniquely by their abscissae, it follows by means of consequence 3° that the operator T^{**} exists. In view of (2) and consequence 2°, T^{**} coincides with T. So the theorem is proved for the case in which the operator T is closed. Let us assume now that the operator T is not closed but has a closure $\bar{T}$. Then, by what has been proved,

$$(\bar{T})^{**} = \bar{T}.$$

But

$$(\bar{T})^{**} = [(\bar{T})^*]^* = (T^*)^* = T^{**}$$

and, hence,

$$T^{**} = \bar{T},$$

which proves the theorem.

To complete the present paragraph, we prove by the method of graphical representation another remarkable proposition.

THEOREM 2: *If T is a closed linear operator with domain dense in* H, *then the product T^*T is a self-adjoint (and, therefore, positive) operator.*

Proof: First, we note that for $f, g \in \mathrm{D}_{T^*T}$,

$$(T^*Tf, g) = (Tf, Tg) = (f, T^*Tg)$$

and

$$(T^*Tf, f) = (Tf, Tf) \geq 0.$$

Thus, T^*T is a positive operator. Let h be an arbitrary element of H. Since T is closed, relation (1) holds and, therefore, the element $\{h, 0\} \in \mathbf{H}$ has a unique representation in the form

$$\{h, 0\} = \{f_0, Tf_0\} + \boldsymbol{U}\{g_0, T^*g_0\}$$

or equivalently,

$$\{h, 0\} = \{f_0, Tf_0\} + i\{T^*g_0, -g_0\}.$$

It follows that

$$h = f_0 + iT^*g_0, \qquad 0 = Tf_0 - ig_0$$

whence

$$h = (E + T^*T)f_0.$$

Thus, for every $h \in \mathrm{H}$, the equation

$$(E + T^*T)f = h \tag{3}$$

is solvable (uniquely). From this it follows that D_{T^*T} is dense in H. Indeed, let h be any vector which is orthogonal to D_{T^*T}. Now, h can be represented in the form (3). Hence, for each $g \in \mathrm{D}_{T^*T}$,

$$0 = (h, g) = ((E + T^*T)f, g) = (f, (E + T^*T)g).$$

Let $g = f$ to get

$$0 = (f,f) + (f, T^*Tf) = (f,f) + (Tf, Tf).$$

This equation yields $f = 0$, which implies that $h = 0$. Therefore, D_{T^*T} is dense in H. It follows that T^*T and, hence, $E + T^*T$ are symmetric operators. But, according to what has been proved, the range of the operator $E + T^*T$ is the whole space H. Therefore (cf. Section 41), $E + T^*T$ is a self-adjoint operator. Since

$$T^*T = (E + T^*T) - E,$$

T^*T is self-adjoint. This completes the proof. Under the conditions of the theorem, it is proved in the same way that TT^* is a positive self-adjoint operator.

47. Matrix Representations of Unbounded Symmetric Operators

The present paragraph is essentially an extension of Section 26. We assume again that the space H is separable and concern ourselves with the question of the matrix representation of an operator A. The operator A is now unbounded, symmetric and closed.

As in Section 26, we select an orthonormal basis $\{e_k\}_1^\infty$ in H, which is no longer arbitrary but must belong to the dense subset D_A. Let

$$Ae_k = c_k \qquad (k = 1, 2, 3, \ldots) \tag{1}$$

and

$$(Ae_k, e_i) = a_{ik} \qquad (i, k = 1, 2, 3, \ldots). \tag{2}$$

We shall attempt to recover the operator A from its values for the vectors e_k or, what is the same thing, from the matrix (a_{ik}).

With this purpose in mind, we introduce the linear envelope $L = L(e_1, e_2, e_3, \ldots)$ of the set of all vectors e_k $(k = 1, 2, 3, \ldots)$. Let B denote the linear operator defined for e_k, $k \geq 1$, by

$$Be_k = c_k, \qquad (k = 1, 2, 3, \ldots) \tag{3}$$

and extended by linearity to L. Thus, the domain D_B of B is the linear manifold L. Since $\bar{a}_{ik} = a_{ki}$, B is symmetric. In view of (1) and (3), A is a closed extension of B. Therefore, the closure (the minimal closed extension) $\bar{B}$ of B satisfies the relation

$$\bar{B} \subset A.$$

If a closed linear operator has the matrix representation (a_{ik}) with respect to the basis $\{e_k\}_1^\infty$ then this operator must be $\bar{B}$ rather than any of its closed extensions, which also satisfy (1) and (2). It is possible that $\bar{B} = A$. In this case we say that the operator A is represented by the matrix (a_{ik}) with

respect to the basis $\{e_k\}_1^\infty$. A change of basis $\{e_k\}_1^\infty$ changes the matrix (a_{ik}) and the operator $\bar{B}$. Therefore, the question arises: given a closed symmetric operator A is it possible to find an orthonormal basis $\{e_k\}_1^\infty$ with respect to which $\bar{B} = A$, i.e., with respect to which the operator A has a matrix representation? Below (Theorem 3) we answer this question in the affirmative.

DEFINITION: *An orthonormal basis $\{e_k\}_1^\infty$ is called a basis for a matrix representation for a closed symmetric operator A if:*

1 *the elements of this basis belong to* D_A; and

2. *there is a minimal closed linear operator which assumes the value* Ae_k *at* e_k, $k \geqq 1$.

In contrast with Section 26 we have not considered yet the question of the construction of the components of Af from those of f. The following two theorems are devoted to this question.

THEOREM 1: *Let A be a closed symmetric operator; let* $\{e_k\}$ *be an arbitrary orthonormal basis, the elements of which belong to* D_A; *and, finally, let*

$$(Ae_k, e_i) = a_{ik} \qquad (i, k = 1, 2, 3, \ldots).$$

Then the value of Af *for each* $f \in D_A$ *is given by the formulas*

$$(4_1) \qquad Af = \sum_{i=1}^{\infty} y_i e_i,$$

$$(4_2) \qquad y_i = \sum_{k=1}^{\infty} a_{ik} x_k \qquad (i = 1, 2, 3, \ldots)$$

if

$$(5) \qquad f = \sum_{k=1}^{\infty} x_k e_k.$$

Proof: The proof follows from the equations,

$$y_i = (Af, e_i) = (f, Ae_i) = \sum_{k=1}^{\infty} (f, e_k)(e_k, Ae_i) =$$

$$= \sum_{k=1}^{\infty} (f, e_k)(Ae_k, e_i) = \sum_{k=1}^{\infty} a_{ik} x_k \quad (i = 1, 2, 3, \ldots).$$

THEOREM 2: *Let A be a closed symmetric operator; let* $\{e_k\}_1^\infty$ *be a basis for a matrix representation of A; and let*

$$a_{ik} = (Ae_k, e_i) \qquad (i, k = 1, 2, 3, \ldots).$$

Finally, define the operator T by the relations

$$Tf = \sum_{i=1}^{\infty} z_i e_i,$$

$$z_i = \sum_{k=1}^{\infty} a_{ik} x_k$$

on the set D_T *of all vectors*

$$f = \sum_{k=1}^{\infty} x_k e_k,$$

for which

$$\sum_{i=1}^{\infty} |\sum_{k=1}^{\infty} a_{ik}\, x_k\,|^2 < \infty.$$

Then $T = A^*$, *i.e.*, T *is the adjoint of* A.

Proof: We prove first that

$$A^* \subset T. \tag{6_1}$$

Let $g \in D_{A^*}$ and $A^*g = g^*$. Letting

$$g = \sum_{k=1}^{\infty} x_k e_k, \qquad g^* = \sum_{i=1}^{\infty} z_i e_i,$$

we obtain

$$z_i = (g^*, e_i) = (g, Ae_i) = \sum_{k=1}^{\infty} (g, e_k)(e_k, Ae_i) =$$

$$= \sum_{k=1}^{\infty} (g, e_k)(Ae_k, e_i) = \sum_{k=1}^{\infty} a_{ik} x_k.$$

We also have

$$\sum_{i=1}^{\infty} |\, z_i\,|^2 = \|\, g^*\,\|^2 < \infty.$$

Therefore, the vector g belongs to D_T and $Tg = g^*$, so that relation (6_1) is proved. In the proof we did not use the hypothesis that the basis $\{e_k\}_1^{\infty}$ is a basis for the matrix representation of A. We now prove that

$$T \subset A^*. \tag{6_2}$$

Letting $g \in D_T$ and

$$g = \sum_{k=1}^{\infty} x_k e_k$$

we obtain

$$(Ae_i, g) = \sum_{k=1}^{\infty} \bar{x}_k (Ae_i, e_k) = \sum_{k=1}^{\infty} a_{ki} \bar{x}_k.$$

But since

$$(Tg, e_i) = \sum_{k=1}^{\infty} a_{ik} x_k = \sum_{k=1}^{\infty} \bar{a}_{ki} x_k$$

we also obtain

$$(Ae_i, g) = \overline{(Tg, e_i)} = (e_i, Tg).$$

It follows that

$$(Af, g) = (f, Tg) \tag{7}$$

for each f in the linear envelope of the vectors e_i $(i = 1, 2, 3, \ldots)$. But, since $\{e_k\}_1^{\infty}$ is a basis for a matrix representation of the operator A, equation

(7) holds for $f \in D_A$. Hence, $g \in D_{A^*}$ and $A^*g = Tg$, so that relation (6_2) is established. Since relations (6_1) and (6_2) yield $T = A^*$, the theorem is proved.

We remark that Theorem 2 yields the equation

$$\sum_{i=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{ik}x_k\Big)\bar{y}_i = \sum_{k=1}^{\infty} x_k\,\overline{\Big(\sum_{i=1}^{\infty} a_{ki}y_i\Big)} \tag{8}$$

where it is assumed that the vector $\sum_{k=1}^{\infty} x_k e_k$ belongs to D_A and that the vector $\sum_{i=1}^{\infty} y_i e_i$ belongs to D_{A^*}. Equation (8) can be expressed in the form

$$\sum_{i=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{ik}x_k\Big)\bar{y}_i = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty} a_{ik}\bar{y}_i\Big)x_k.$$

If the reversal of order of summation which this equation indicates is valid for every vector in D_{A^*}, then the operator A^* is symmetric. In this and only this case A is a self-adjoint operator.

The above theorems indicate why matrix representations of symmetric operators cannot be based on formulas of the form (4_1), (4_2) and (5), as was done in Section 26 for bounded operators.

The possibility of the matrix representation of an operator is equivalent to the property that the operator can be recovered from the matrix.

THEOREM 3: *There exists a basis for matrix representation of each closed symmetric operator A.*

Proof: We prove first that there exists a sequence $\{f_k\}_1^{\infty} \subset D_A$ such that, for each $f \in D_A$, there is a subsequence $\{f_{k_i}\}_{i=1}^{\infty}$ for which

$$\lim_{i\to\infty} f_{k_i} = f, \quad \lim_{i\to\infty} Af_{k_i} = Af.$$

After this, in order to complete the proof of the theorem it remains only to orthogonalize the sequence $\{f_k\}_1^{\infty}$. As a preliminary to the construction of the sequence $\{f_k\}_1^{\infty}$, we choose an arbitrary sequence $\{h_k\}_1^{\infty}$ which is dense in H. If for a triple of integers, (m, n, p), there exist elements $f \in D_A$ satisfying the inequalities

$$\|h_m - f\| \leqq \frac{1}{p}, \qquad \|h_n - Af\| \leqq \frac{1}{p},$$

then we associate with this triple any one such element f and denote it by $f_{m,n,p}$. Since $\{h_k\}_1^{\infty}$ is dense in H, this defines an infinite sequence $\{f_{m,n,p}\}$. The possibility that a triple (m, n, p) might not have a corresponding element, or that the same element might correspond to different triples will not matter in what follows. We enumerate the elements $f_{m,n,p}$ in order to obtain the desired sequence $\{f_k\}_1^{\infty}$. For the proof we choose an arbitrary

element of $f \in D_A$, an arbitrary number $\varepsilon > 0$, and any integer $p' \geqq \frac{2}{\varepsilon}$. Since the sequence $\{h_k\}_1^\infty$ is dense in H there exist integers m', n' such that

$$(9_1) \qquad \|h_{m'} - f\| \leqq \frac{1}{p'}, \quad \|h_{n'} - Af\| \leqq \frac{1}{p'}.$$

In view of (9_1) the triple (m', n', p') has an associated element $f_{m',n',p'}$ such that

$$(9_2) \qquad \|h_{m'} - f_{m',n',p'}\| \leqq \frac{1}{p'}, \quad \|h_{n'} - Af_{m',n',p'}\| \leqq \frac{1}{p'}.$$

From (9_1) and (9_2) it follows that

$$\|f_{m',n',p'} - f\| \leqq \varepsilon, \quad \|Af_{m',n',p'} - Af\| \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the existence of the required subsequence $\{f_{k_i}\}_{i=1}^\infty$ is established, so that the theorem is proved.

We have proved that to each closed symmetric operator there corresponds a (Hermitian) matrix which represents the operator in terms of some basis. However, not every Hermitian matrix defines a symmetric operator.

THEOREM 4: *If the Hermitian matrix* (a_{ik}) *satisfies the relations*

$$(10) \qquad \sum_{i=1}^{\infty} |a_{ik}|^2 < \infty \qquad (k = 1, 2, 3, \ldots),$$

then, in terms of each orthonormal basis, it defines a closed symmetric operator.

Proof: It suffices to let

$$Ae_k = \sum_{i=1}^{\infty} a_{ik} e_i \quad (k = 1, 2, 3, \ldots)$$

and then to construct the operator $\bar{B}$ by the method indicated at the beginning of the present section.

The Hermitian matrices satisfying condition (10), but which are not bounded, are called *unbounded* Hermitian matrices. An unbounded Hermitian matrix (a_{ik}) does not admit, in general, a transformation by an arbitrary unitary matrix by means of the formula

$$(\mathring{a}_{ik}) = (u_{ir}^*) \cdot (a_{rs}) \cdot (u_{sk}).$$

If the corresponding infinite series converge so that such a transformation is formally possible, then it can happen, nevertheless, that the transformed Hermitian matrix $(\mathring{a}_{ik})$ does not satisfy condition (10) and hence does not determine an operator in H. Moreover, even if the transformed matrix satisfies condition (10) then the operator $\mathring{A}$ determined by it may not coincide with A. (It is remarkable that the intersection $D_A \cap D_{\mathring{A}}$ may turn out to be empty.)

These various possibilities form the foundations for the so-called pathological properties of unbounded Hermitian matrices.[6] These properties make matrices unsuitable for the study of unbounded symmetric operators.

48. The Operation of Multiplication by the Independent Variable

If $[a, b]$ is a finite interval, then the operator Q of multiplication by independent variable is defined for each function $\varphi = \varphi(t) \in L^2(a, b)$ by the equation

$$Q\varphi(t) = t\varphi(t). \tag{1}$$

In this case Q is a bounded self-adjoint operator, the norm of which is equal to the larger of the numbers $|a|$ and $|b|$. If $[a, b]$ is an infinite interval, then the multiplication operator Q is defined by formula (1) on the manifold D_Q of functions $\varphi(t) \in L^2(a, b)$ such that $t\varphi(t) \in L^2(a, b)$. This manifold D_Q is dense in $L^2(a, b)$ since it contains the set D of all functions in $L^2(a, b)$ each of which vanishes outside of a finite interval (the interval depending on the function). If $[a, b]$ is an infinite interval then it is evident that Q is an unbounded symmetric operator. We show now that Q is self-adjoint.

Let $\psi \in D_{Q^*}$ and $\psi^* = Q^*\psi$. Then for each $\varphi \in D_Q$,

$$(Q\varphi, \psi) = (\varphi, \psi^*).$$

Equivalently,

$$\int_a^b t\varphi(t)\overline{\psi(t)}\,dt = \int_a^b \varphi(t)\overline{\psi^*(t)}\,dt$$

or

$$\int_a^b \varphi(t)\{t\overline{\psi(t)} - \overline{\psi^*(t)}\}\,dt = 0.$$

The last equation holds, in particular, for any function $\varphi(t)$ in $L^2(a, b)$ which vanishes outside of a finite interval. Therefore, for any finite α and β in the interval (a, b),

$$\int_\alpha^\beta \varphi(t)\{t\overline{\psi(t)} - \overline{\psi^*(t)}\}dt = 0.$$

It follows that, for almost all t in $[a, b]$,

$$\psi^*(t) = t\psi(t).$$

Hence

$$t\psi(t) \in L^2(a, b),$$

[6] Cf. J. v. Neumann [1].

and

$$\psi \in D_Q,\ \psi^* = Q\psi.$$

This argument shows that $Q^* \subset Q$. But since $Q \subset Q^*$ by the symmetry of Q, we have $Q = Q^*$. From the equation $Q = Q^*$ it follows, in particular, that Q is closed.

If we restrict the multiplication operator to a smaller domain, i.e., if we define an operator Q_1 by

$$Q_1\varphi(t) = t\varphi(t),\quad D_{Q_1} = D,$$

where $D \subset D_Q$, then the above argument yields

$$Q_1^* = Q.$$

It follows that (cf. Section 46)

$$Q = Q^* = Q^{**} = \bar{Q}_1,$$

i.e., the operator Q defined originally is the closure of the operator Q_1.

The operator of multiplication by the independent variable does not have any eigenfunctions. Since

$$Q\varphi = \lambda\varphi$$

implies that

$$\int_a^b |t-\lambda|^2 |\varphi(t)|^2 dt = 0$$

we have $\varphi(t) = 0$ almost everywhere, i.e., $\varphi = 0$.

All points of the interval $[a, b]$ belong to the continuous spectrum of the operator Q since the manifold

$$\Delta_Q(\lambda) = (Q - \lambda E) D_Q \qquad (a < \lambda < b)$$

consists of all functions $\psi(t)$ in $L^2(a, b)$ which remain in $L^2(a, b)$ after division by $t - \lambda$; i.e., $\psi \in \Delta_Q(\lambda)$ if and only if

$$\psi(t) \in L^2(a, b)$$

and

$$\frac{\psi(t)}{t-\lambda} \in L^2(a, b).$$

It is evident that $\Delta_Q(\lambda)$ is dense in $L^2(a, b)$. But it does not coincide with $L^2(a, b)$ because, for instance, it does not contain a function equal to one in a neighborhood of the point $t = \lambda$.

The subspace $M = M_e$ of functions in $L^2(a, b)$ which are equal to zero outside of some fixed point set $e \subset [a, b]$ evidently reduces[7] the operator Q. If $\varphi \in L^2(\alpha, \beta)$, where $\beta - \alpha \leqq \varepsilon$, then the operator Q satisfies the inequality,

[7] It is not difficult to prove that all subspaces reducing the operator Q have this form.

$$\| Q\varphi - \lambda_0 \varphi \| \leq \varepsilon \| \varphi \|,$$

for each fixed λ_0 in the interval $[\alpha, \beta]$; i.e., Q differs by not more than ε in norm from the similarity transformation $\lambda_0 E$. Thus, decomposing the interval $[a, b]$ into subintervals of sufficiently small length, we get a decomposition of the space $L^2(a, b)$ into a countable sum of subspaces, each reducing Q and in each of which the operator Q differs by an arbitrarily small amount from a similarity operator.[8]

It is possible to consider the operator of multiplication not only by the independent variable but also by a function of it. We restrict ourselves here to one particular case. Let the operator U be defined on all functions $\varphi(t)$ in $L^2(0, 2\pi)$ by the equation

$$U\varphi(t) = e^{it}\varphi(t).$$

Each element of the orthonormal basis $\left\{\frac{1}{\sqrt{2\pi}} e^{ikt}\right\}_{k=-\infty}^{\infty}$ is transformed by the operator U into the following element:

$$Ue^{ikt} = e^{i(k+1)t} \qquad (-\infty < k < \infty).$$

From this fact it follows that the operator U is unitarily equivalent to the operator U_0 defined in a separable space H in Section 40. Indeed, if we define the isometric operator V which maps H onto $L^2(0, 2\pi)$ by the equation

$$Ve_k = \frac{1}{\sqrt{2\pi}} e^{ikt} \quad (-\infty < k < \infty), \tag{2}$$

then

$$U = V U_0 V^{-1}.$$

This representation of U in terms of U_0 will be used to find the (infinitely many) subspaces which reduce U_0.[9] We shall base the derivation on the following general proposition (the proof of which is left to the reader). If the subspace G of the space H reduces a linear operator T, and if the operator T_1 defined on the space H_1 is unitarily equivalent to the operator T ($T_1 = V T V^{-1}$), then the subspace $G_1 = VG$ of the space H_1 reduces the operator T_1. According to this proposition the class of subspaces reducing the operator U_0 includes the set G_e, which consists of all the elements

$$f(t) = V^{-1}\varphi(t)$$

such that $\varphi(t) = 0$ outside of the arbitrary subset e of $[0, 2\pi]$ and the operator V is defined by formula (2).

[8] In Chapter VI we shall prove that every self-adjoint operator has an analogous approximate decomposition into a similarity transformation. This fact plays a fundamental role in all of the theory.

[9] Cf. the end of Section 40.

In conclusion we note that, in place of the multiplication operator Q in the space $L^2(a, b)$, one can consider the operator Q_σ of multiplication by an independent variable in the space $L^2_\sigma(a, b)$. It is not difficult to verify that Q_σ is a self-adjoint operator. We propose to the reader as useful exercises the proofs of the following propositions.

(a) The real regular values of the operator Q_σ are the points of constancy of the function $\sigma(t)$.

(b) The eigenvalues of the operator Q_σ are the points of discontinuity of the function $\sigma(t)$.

(c) The continuous spectrum of the operator Q_σ is the set of all non-isolated points of increase of the function $\sigma(t)$.

The operators Q_σ play a special role in the theory of self-adjoint operators: in Chapter VI we shall see that the study of any self-adjoint operator can be reduced to the study of the operators Q_σ.

49. A Differential Operator

We shall consider the *differential operator* P in $L^2(a, b)$ which is defined by the equation

$$P\varphi = i\frac{d\varphi}{dt}$$

for each function $\varphi(t)$ in its domain D_P. A function $\varphi(t)$ belongs to the domain D_P if certain conditions, given below, are satisfied.

(A) If $\varphi \in D_P$ then the function $\varphi(t)$ is absolutely continuous on each finite subset of the interval $[a, b]$, and both $\varphi(t)$ and $\varphi'(t)$ belong to $L^2(a, b)$. We examine separately the cases of a finite interval [a, b], a semi-axis, and the whole axis. In the first two cases D_P consists of the functions which, besides condition (A), also satisfy certain boundary conditions (B).

1° *Finite interval.* In this case we assume that the interval is $[0, 2\pi]$ and the boundary conditions have the form

(B) $$\varphi(0) = \varphi(2\pi) = 0.$$

The set D_P of functions satisfying conditions (A) and (B) is evidently dense in $L^2(0, 2\pi)$. Then P is a symmetric (unbounded) operator, since

$$(P\varphi,\psi) = \int_0^{2\pi} i\varphi'(t)\overline{\psi(t)}dt$$

$$= i\{\varphi(2\pi)\overline{\psi(2\pi)} - \varphi(0)\overline{\psi(0)} + \int_0^{2\pi} \varphi(t)\{\overline{i\psi'(t)}\}\,dt$$

for $\varphi, \psi \in D_P$. This equation implies that

$$(P\varphi, \psi) = (\varphi, P\psi),$$

since the integral expression vanishes. The integral expression vanishes also in case φ belongs to D_P and ψ satisfies only the condition (A). Hence, every function $\psi(t)$ satisfying condition (A) belongs to D_{P^*}, and

$$P^*\psi(t) = i\psi'(t).$$

Conversely, let $\psi \in D_{P^*}$ and $P^*\psi = \psi^*$. Then, for each $\varphi \in D_P$,

$$(P\varphi, \psi) = (\varphi, \psi^*) = \int_0^{2\pi} \varphi(t)\overline{\psi^*(t)}\,dt =$$

$$= -i\int_0^{2\pi} \varphi(t)\,\frac{d}{dt}\overline{\left\{-\int_0^t i\psi^*(s)ds + C\right\}}\,dt,$$

where C is an arbitrary constant. Integrating by parts we get

$$(1) \qquad (P\varphi, \psi) = \int_0^{2\pi} i\varphi'(t)\overline{\left\{-\int_0^t i\psi^*(s)\,ds + C\right\}}\,dt,$$

since $\varphi(t)$ vanishes at the end-points of the integral. From (1) it follows that, for $\varphi(t) \in D_P$,

$$(2) \qquad \int_0^{2\pi} \varphi'(t)\overline{\left\{\psi(t) + \int_0^t i\psi^*(s)\,ds - C\right\}}\,dt = 0.$$

We define C by the equation

$$\int_0^{2\pi}\left\{\psi(t) + \int_0^t i\psi^*(s)\,ds - C\right\}dt = 0$$

and substitute for $\varphi(t)$ the function

$$\varphi_0(t) = \int_0^t\left\{\psi(t) + \int_0^t i\psi^*(s)\,ds - C\right\}dt,$$

which evidently belongs to D_P. Then (2) assumes the form

$$\int_0^{2\pi} |\,\psi(t) + \int_0^t i\psi^*(s)\,ds - C\,|^2\,dt = 0$$

which yields

$$\psi(t) + \int_0^t i\psi^*(s)\,ds - C = 0.$$

Therefore, for almost all t,

$$i\psi'(t) = \psi^*(t).$$

We have proved that the domain of the operator P^* is the set of all functions $\psi(t)$ satisfying relation (A), and that

$$P^*\psi(t) = i\psi'(t).$$

From the proof of these facts it follows that the symmetric operator P is not self-adjoint; in fact, the functions in D_P must satisfy two conditions (A) and (B), whereas the functions in D_{P^*} must satisfy only the condition (A).

We prove now that the operator P is closed. In place of a direct proof, we proceed, first, to prove that (cf. Section 46)

$$P^{**} = P.$$

From the relation

$$P \subset P^*$$

it follows that

$$P^{**} \subset P^*$$

Therefore the functions $\chi(t)$ in $D_{P^{**}}$ satisfy condition (A) and

$$P^{**}\chi(t) = i\chi'(t).$$

Hence, for each $\psi \in D_{P^*}$,

$$\int_0^{2\pi} \psi(t)\,\overline{i\chi'(t)}\,dt = (\psi, P^{**}\chi) = (P^*\psi, \chi) = \int_0^{2\pi} i\psi'(t)\,\overline{\chi(t)}\,dt =$$

$$= i[\psi(2\pi)\,\overline{\chi(2\pi)} - \psi(0)\,\overline{\chi(0)}] + \int_0^{2\pi} \psi(t)\,\overline{i\chi'(t)}\,dt,$$

which implies that

$$\psi(2\pi)\,\overline{\chi(2\pi)} - \psi(0)\,\overline{\chi(0)} = 0.$$

Since the values of $\psi(0)$ and $\psi(2\pi)$ are arbitrary, this equation is satisfied if and only if

$$\chi(0) = \chi(2\pi) = 0,$$

i.e., if $\chi(t) \in D_P$. We have proved that

$$P^{**} \subset P.$$

Since the reverse relation

$$P \subset P^{**}$$

always holds, we have

$$P = P^{**}.$$

This equation implies that the operator P is closed.

If we replace (A) and (B) by more rigid conditions, requiring, for example, that the functions in the domain of the operator be repeatedly

(or even infinitely often) differentiable and that these derivatives vanish for $t = 0$ and $t = 2\pi$, then the closure of the operator is P. In order to be convinced of this fact one may verify that the adjoint of the adjoint of the new operator coincides with P.

However, it is possible to strengthen (A) and (B) to such an extent that the closure of the operator obtained does not coincide with the operator P. For example, if we leave condition (A) unchanged, and replace condition (B) by

$$\varphi(0) = \varphi(\pi) = \varphi(2\pi) = 0,$$

then the operator P_0 obtained turns out to be closed, but $P_0 \neq P$ (of course, $P_0 \subset P$).

In this connection we remark that the subspace of functions in $L^2(0, 2\pi)$ which vanish for $\pi < t \leqq 2\pi$ (we can identify this subspace with $L^2(0, \pi)$) reduces the operator P_0 but does not reduce the operator P.

We now look for the symmetric extensions of the operator P. Let $\tilde{P}$ be one of them. Since $\tilde{P} \subset P^*$ the functions in $D_{\tilde{P}}$ satisfy relation (A). Hence, for all functions $\varphi, \psi \in D_{\tilde{P}}$ we have

$$(\tilde{P}\varphi, \psi) = \int_0^{2\pi} i\varphi'(t)\,\overline{\psi(t)}\,dt = i[\varphi(2\pi)\overline{\psi(2\pi)} - \varphi(0)\,\overline{\psi(0)}] + \int_0^{2\pi} \varphi(t)\overline{i\psi'(t)}\,dt =$$

$$= i[\varphi(2\pi)\,\overline{\psi(2\pi)} - \varphi(0)\,\overline{\psi(0)}] + (\varphi, \tilde{P}\psi).$$

Since the operator $\tilde{P}$ is symmetric, it must satisfy the relation

$$\varphi(2\pi)\,\overline{\psi(2\pi)} - \varphi(0)\,\overline{\psi(0)} = 0. \tag{3}$$

Since $\tilde{P} \neq P$, there exists a function $\psi_0(t)$ in $D_{\tilde{P}}$ not satisfying relation (B); assume, for the sake of definiteness, that $\psi_0(2\pi) \neq 0$. Letting $\psi(t) = \psi_0(t)$ in (3), we obtain for each function $\varphi(t) \in D_{\tilde{P}}$ the relation

$$\varphi(2\pi) = \theta\,\varphi(0) \tag{$\tilde{\text{B}}$}$$

where

$$\theta = \frac{\overline{\psi_0(0)}}{\overline{\psi_0(2\pi)}}.$$

Since condition ($\tilde{\text{B}}$) holds for $\varphi(t) = \psi_0(t)$, we must have $|\theta| = 1$.

Our result may be stated as follows: all functions $\varphi(t)$ in $D_{\tilde{P}}$ must satisfy conditions (A) and ($\tilde{\text{B}}$), where for each extension $\tilde{P}$ the constant θ has absolute value one.

We now prove the converse: every function $\psi(t)$ which satisfies conditions (A), ($\tilde{\text{B}}$) belongs to $D_{\tilde{P}}$. With this purpose in mind we select a constant α such that

$$\psi(0) - \alpha\,\psi_0(0) = 0$$

and we let

$$\varphi(t) = \psi(t) - a\psi_0(t).$$

It is easy to see that $\varphi(t)$ satisfies conditions (A) and (B) and, hence, belongs to D_P. But, since $D_P \subset D_{\tilde{P}}$, we have $\varphi(t) \in D_{\tilde{P}}$, so that the function

$$\psi(t) = \varphi(t) + a\psi_0(t)$$

also belongs to $D_{\tilde{P}}$.

Thus, the symmetric extensions $\tilde{P}$ of the operator P are characterized by conditions (A) and ($\tilde{\text{B}}$). The extension of the operator P has led to a weakening of condition (B).

In view of the fact that each extension is determined by a number θ ($|\theta| = 1$), which appears in ($\tilde{\text{B}}$), we shall write P_θ instead of $\tilde{P}$. It is not difficult to verify that $D_{P_\theta^*}$ consists of the functions $\psi(t)$ in D_{P^*} such that

$$\varphi(2\pi)\overline{\psi(2\pi)} - \varphi(0)\overline{\psi(0)} = 0$$

for each $\varphi(t) \in D_{P_\theta}$. Hence it follows that D_{P_θ} and $D_{P_\theta^*}$ coincide, so that every extension P_θ of the operator P is a self-adjoint operator.

We now determine the spectrum of the operator P_θ (for simplicity only for $\theta = 1$). The equation

$$P_1\varphi = \lambda\varphi$$

implies that

$$i\varphi'(t) = \lambda\varphi(t)$$

and

$$\varphi(2\pi) = \varphi(0).$$

Hence

$$\lambda = \lambda_k = k$$
$$\varphi(t) = \varphi_k(t) = e^{-ikt} \qquad (\pm k = 0, 1, 2, \ldots).$$

It is not difficult to verify that for $\lambda \neq \lambda_k$ the equation

$$(P_1 - \lambda E)f = g$$

or, equivalently, the equation

$$if'(t) - \lambda f(t) = g(t) \qquad [f(2\pi) = f(0)]$$

is solvable for each function $g(t) \in L^2(0, 2\pi)$. Hence, P_1 has no continuous spectrum.

2° *Semi-axis* $[0, \infty)$. If the function $\varphi(t)$ satisfies condition (A) in the case of a semi-axis, then the product $\varphi(t)\varphi'(t)$ is absolutely integrable on this interval. The formula

$$\int_0^t \varphi(s)\overline{\varphi'(s)}\,ds = |\varphi(t)|^2 - |\varphi(0)|^2 - \int_0^t \varphi'(s)\overline{\varphi(s)}\,ds$$

shows that $|\varphi(t)|$ has a limit as $t \to \infty$. But, since $\varphi(t) \in L^2(0, \infty)$, we must have

$$\lim_{t\to\infty} \varphi(t) = 0.$$

As we see, the boundary condition on the right end of the semi-axis is automatically fulfilled.

For the second condition in the definition of the domain D_P we shall require that

(B) $$\varphi(0) = 0.$$

For every $\varphi, \psi \in D_P$ we have

$$(P\varphi, \psi) = i\int_0^\infty \varphi'(t)\,\overline{\psi(t)}\,dt = \int_0^\infty \varphi(t)\,\overline{i\psi'(t)}\,dt = (\varphi, P\psi).$$

Hence P is a symmetric operator.

As in the case of a finite interval it is not difficult to show that D_{P^*} is the set of all functions satisfying the single condition (A), and that

$$P^*\psi = i\psi'.$$

Thus, $P \neq P^*$, so that the operator P is not self-adjoint.

In contrast with the case of a finite interval the differential operator on the semi-axis does not have a symmetric proper extension. In fact the domain of definition of such an extension $\tilde{P}$ would have to contain a function $\varphi_0(t)$ which is different from zero for $t = 0$. But then we would have

$$(\tilde{P}\psi_0, \psi_0) = i\int_0^\infty \psi_0'(s)\,\overline{\psi_0(s)}\,ds = i\,|\psi_0(0)|^2 + \int_0^\infty \psi_0(s)\,i\overline{\psi_0'}(s)\,ds \neq (\psi_0, \tilde{P}\psi_0)$$

which is impossible. Thus, the differential operator on the semi-axis is a maximal symmetric operator. Later (cf. Section 82) we shall prove that it is irreducible.

3° *The whole real axis.* For each function $\varphi(t)$ satisfying condition (A) the boundary conditions

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to-\infty} \varphi(t) = 0$$

are automatically fulfilled. Therefore, the domain of D_P is defined by the single requirement (A) and it is proved without difficulty that P is a self-adjoint operator. The operator P has no eigenvalues because the equation

$$i\frac{d\varphi}{dt} = \lambda\varphi$$

does not have a nontrivial solution in $L^2(-\infty, \infty)$.

We now establish a connection between the operator Q (of multiplication by the independent variable) and the operator P. From this con-

nection it will follow that every point of the real axis belongs to the continuous spectrum of the operator P. The connection between the two operators amounts to the fact that they are unitarily equivalent. To show this we shall use the relations

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(s)\, e^{-ist}\, ds,$$

$$i\psi'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s\, \varphi(s)\, e^{-ist}\, ds.$$

Thus, multiplication of the function $\varphi(s)$ by s corresponds to differentiation of the function $\psi(t)$.

THEOREM: *The operators P and Q are related by*

$$P = \mathfrak{F}\, Q\, \mathfrak{F}^{-1}$$

where $\mathfrak{F}$ is the Fourier-Plancherel operator.

Proof: The proof consists of two parts. First, we shall show that $h \in \mathrm{D}_Q$ implies that

$$\mathfrak{F}h \in \mathrm{D}_P, \quad P\mathfrak{F}h = \mathfrak{F}Qh$$

and then we shall show that $g \in \mathrm{D}_P$ implies that

$$\mathfrak{F}^{-1}g \in \mathrm{D}_Q, \quad \mathfrak{F}^{-1}Pg = Q\mathfrak{F}^{-1}g.$$

Let $h \in \mathrm{D}_Q$. Then

$$(4) \quad \mathfrak{F}\, Q\, h(t) = \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ist} - 1}{-is}\, s h(s)\, ds = i \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{e^{-ist} - 1\}\, h(s)\, ds$$

and, since $h \in \mathrm{D}_Q$,

$$\int_{-\infty}^{\infty} |h(s)|\, ds < \infty.$$

Therefore,

$$\mathfrak{F}h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} h(s)\, ds,$$

and relation (4) can be expressed in the form

$$\mathfrak{F}\, Q\, h(t) = i \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} h(s)\, ds$$

which implies both the inclusion $\mathfrak{F}\, h \in \mathrm{D}_P$ and the equation

$$\mathfrak{F}Qh = P\mathfrak{F}h.$$

Let us assume now that $g \in D_P$. Then

$$\mathfrak{F}^{-1}Pg(t) = \frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ist}-1}{is}\,ig'(s)\,ds =$$

$$= -\frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{e^{ist}ist-(e^{ist}-1)}{s^2}\right)g(s)\,ds =$$

$$= \frac{d}{dt}\frac{t}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ist}-1}{is}g(s)\,ds + \frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ist}-1-ist}{s^2}g(s)\,ds =$$

$$= t\frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ist}-1}{is}g(s)\,ds = t\,\mathfrak{F}^{-1}g(t).$$

Since the left member belongs to $L^2(-\infty, \infty)$, so does the right member. Hence, $\mathfrak{F}^{-1}g \in D_Q$ and

$$\mathfrak{F}^{-1}Pg = Q\mathfrak{F}^{-1}g.$$

Since every real point belongs to the continuous spectrum of Q, the equation

$$(Q-\lambda E)g = f$$

is not always solvable. Hence, the equation,

$$(Q-\lambda E)\,\mathfrak{F}^{-1}h = f,$$

is not always solvable. Neither is

$$\mathfrak{F}(Q-\lambda E)\,\mathfrak{F}^{-1}h = \mathfrak{F}f$$

which can be reduced to the form

$$(P-\lambda E)h = f_1.$$

Since P has no eigenvalues, every real point belongs to its continuous spectrum.

Using the unitary equivalence of the operators Q and P it is easy to determine the subspaces which reduce P. Each such subspace consists of the functions

$$\psi(t) = \mathfrak{F}\varphi(t)$$

where $\varphi(t)$ is equal to zero outside of some fixed set on the real axis.

50. The Inversion of Singular Integrals

To begin with let us assume the following important theorem[10] from the theory of functions: *for each function* $g(x) \in L^2(-\infty, \infty)$, *the equation*

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty}{}' \frac{f(y)}{x-y}\, dy \tag{1}$$

(where the prime signifies that the integral is considered as the principal value in the sense of Cauchy) has a solution[11]; *the solution belongs to* $L^2(-\infty, \infty)$, *and is given by the formula*

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty}{}' \frac{g(y)}{y-x}\, dy. \tag{2}$$

Moreover,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |g(x)|^2 dx.$$

Therefore, the singular integrals in (1) *and* (2) *define unitary operators in* $L^2(-\infty, \infty)$.

Equations (1) and (2) form a pair of inversion formulas. If we assume that the first of the functions $f(x)$ and $g(x)$ is even, we find without difficulty that the second is odd. This leads to the following pair of inversion formulas in the space $L^2(0, \infty)$:

$$g(x) = \frac{2}{\pi} \int_{0}^{\infty}{}' \frac{f(y)\, x}{x^2 - y^2}\, dy,$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty}{}' \frac{g(y)\, y}{y^2 - x^2}\, dy$$

where

$$\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |g(x)|^2 dx.$$

Letting

$$x^2 = s,\ y^2 = t,\ f(x) = F(s),\ g(x) = \sqrt{s}\, G(s),$$

[10] Cf. Titchmarsh [1], page 120.

[11] When we say that a function is the solution of the equation (1) we mean that, after the substitution of this function, the right member exists and equals the left member for almost all x.

we get

$$(3)\qquad \begin{cases} G(s)=\dfrac{1}{\pi}\displaystyle\int_0^{\infty}{}' \dfrac{1}{\sqrt{t}}\,\dfrac{F(t)}{s-t}\,dt, \\ F(s)=\dfrac{1}{\pi}\displaystyle\int_0^{\infty}{}' \sqrt{t}\,\dfrac{G(t)}{t-s}\,dt. \end{cases}$$

Therefore,

$$(4)\qquad \int_0^{\infty}\frac{1}{\sqrt{t}}\,|F(t)|^2\,dt=\int_0^{\infty}\sqrt{t}\,|G(t)|^2 dt.$$

If we agree to denote the space $L_\sigma^2(a, b)$, where $\sigma'(x)=p(x)$, by the symbol $L^2(p(x); a, b)$, then we can say that formulas (3) give an isometric mapping of $L^2\left(\frac{1}{\sqrt{x}}; 0, \infty\right)$ onto $L^2(\sqrt{x}; 0, \infty)$.

Now we can easily derive inversion formulas for a finite interval. With this aim in mind we let

$$s=\frac{1+x}{1-x},\quad t=\frac{1+y}{1-y}$$

in (3) and (4), to get the formulas

$$(5)\qquad \begin{cases} g(x)=\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}{}' \sqrt{\dfrac{1-y}{1+y}}\,\dfrac{f(y)}{x-y}\,dy, \\ f(x)=\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}{}' \sqrt{\dfrac{1+y}{1-y}}\,\dfrac{g(y)}{y-x}\,dy, \end{cases}$$

$$(6)\qquad \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,|f(x)|^2\,dx=\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,|g(x)|^2\,dx.$$

This gives an isometric mapping of the space

$$L^2\left(\sqrt{\frac{1-x}{1+x}}; -1, 1\right)$$

onto the space

$$L^2\left(\sqrt{\frac{1+x}{1-x}}; -1, 1\right).$$

These formulas lead to an inversion in which, instead of one interval of the real axis, there is a system of such intervals. We confine ourselves here to one simple example.[12] Let E denote a point set which consists of the finite intervals

$$(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n) \quad (a_1 < b_1 < a_2 < b_2 < \ldots < b_n)$$

and let

$$p(x) = \sqrt{-\frac{(x-b_1)(x-b_2)\ldots(x-b_n)}{(x-a_1)(x-a_2)\ldots(x-a_n)}}, \quad q(x) = \frac{1}{p(x)}.$$

Then the desired formulas have the form

$$\begin{cases} g(x) = \dfrac{1}{\pi} \displaystyle\int_E{}' p(y) \frac{f(y)}{y-x}\, dy, \\ f(x) = \dfrac{1}{\pi} \displaystyle\int_E{}' q(y) \frac{g(y)}{x-y}\, dy, \end{cases}$$

and

$$\int_E p(x) |f(x)|^2 dx = \int_E q(x) |g(x)|^2 dx.$$

The isometric mapping of the space $L^2(p(x); E)$ on the space $L^2(q(x); E)$, defined by these formulas, has a curious property. It maps polynomials into polynomials of the same power. The proofs and further generalizations and examples the reader will find in the works cited above.

[12] N. I. Akhieser [1], N. I. Muschelishvili [1].

Chapter V

SPECTRAL ANALYSIS OF COMPLETELY CONTINUOUS OPERATORS

51. A Lemma

The present chapter is devoted to the spectral theory of certain classes of completely continuous operators. Since it is a direct and easily surveyed generalization of the corresponding sections of linear algebra and of the elementary theory of integral equations, the spectral theory of completely continuous operators represents the most natural introduction to the general spectral theory of operators in Hilbert space.

In the formulation of the spectral theory of completely continuous operators, the completeness of the space, as we shall see below, is not generally exploited. Moreover, by the elimination of the requirement of completeness, the domain of application of the theory is extended. Therefore, in the present chapter, together with propositions referring to operators in a Hilbert space H, there is established a series of propositions concerning operators in an arbitrary inner product space R. A number of these propositions are related to the lemma to which the present section is devoted.

LEMMA: *If* $\{g_k\}_0^\infty$ *is an infinite orthonormal sequence of vectors in* R *and*

$$Ag_k = \beta_{k0}g_0 + \beta_{k1}g_1 + \ldots + \beta_{kk}g_k \qquad (k = 1, 2, 3, \ldots),$$

where A is a completely continuous operator in R, *then*

$$\lim_{k\to\infty} \beta_{kk} = 0.$$

Proof: Let $n > m$. Then

$$\|Ag_n - Ag_m\|^2 = \|\beta_{nn}g_n + \ldots + \beta_{n,m+1}g_{m+1} + \\ + (\beta_{nm} - \beta_{mm})g_m + \ldots + (\beta_{n0} - \beta_{m0})g_0\|^2 = \\ = |\beta_{nn}|^2 + \ldots + |\beta_{n,m+1}|^2 + |\beta_{nm} - \beta_{mm}|^2 + \ldots + |\beta_{n0} - \beta_{m0}|^2 \geqq |\beta_{nn}|^2.$$

Suppose that β_{kk} does not tend to zero as $k \to \infty$. Then there exists an infinite sequence of positive integers

$$n_1 < n_2 < n_3 < \ldots,$$

such that

$$|\beta_{n_j n_j}| \geqq \delta > 0 \qquad (j = 1, 2, 3, \ldots)$$

for some δ Therefore,

$$\| Ag_{n_k} - Ag_{n_i} \|^2 \geqq \delta^2 > 0$$

which implies that the infinite sequence of vectors $\{Ag_{n_j}\}_{j=1}^{\infty}$ does not contain a convergent subsequence, and this contradicts the compactness of the set of vectors $\{Ag_k\}_0^{\infty}$; i.e., it contradicts the complete continuity of the operator A.

52. Properties of the Eigenvalues of Completely Continuous Operators in R

The propositions which we establish in the present and following sections are generalizations of well-known propositions in the theory of integral equations. In the present section, as its title indicates, operators are considered in an inner product space R. In the following section, we shall assume that the system R is complete, so that R becomes a Hilbert space.

THEOREM 1: *If A is a completely continuous operator in* R *and $\rho > 0$, then A has only a finite number of linearly independent eigenvectors such that the corresponding eigenvalues exceed ρ in modulus.*

Proof: Supposing the contrary, we assume that there exists an infinite sequence of linearly independent vectors f_n $(n = 1, 2, 3, \ldots)$, for which

$$Af_n = \lambda_n f_n, \quad |\lambda_n| > \rho > 0 \qquad (n = 1, 2, 3, \ldots).$$

Orthogonalizing $\{f_n\}_1^{\infty}$, we get the orthonormal sequence of vectors

$$\begin{aligned} g_1 &= \alpha_{11} f_1, \\ g_2 &= \alpha_{21} f_1 + \alpha_{22} f_2, \\ &\cdots\cdots\cdots \\ g_k &= \alpha_{k1} f_1 + \alpha_{k2} f_2 + \ldots + \alpha_{kk} f_k, \\ &\cdots\cdots\cdots \end{aligned}$$

Then

$$Ag_k = \alpha_{k1} Af_1 + \alpha_{k2} Af_2 + \ldots + \alpha_{kk} Af_k = \alpha_{k1}\lambda_1 f_1 + \alpha_{k2}\lambda_2 f_2 + \ldots + \alpha_{kk}\lambda_k f_k,$$

and, consequently,

$$\begin{aligned} Ag_k - \lambda_k g_k &= \alpha_{k1}(\lambda_1 - \lambda_k) f_1 + \ldots + \alpha_{k,k-1}(\lambda_{k-1} - \lambda_k) f_{k-1} = \\ &= \beta_{k1} g_1 + \beta_{k2} g_2 + \ldots + \beta_{k,k-1} g_{k-1}. \end{aligned}$$

Thus,

$$Ag_k = \beta_{k1} g_1 + \beta_{k2} g_2 + \ldots + \beta_{k,k-1} g_{k-1} + \lambda_k g_k.$$

By the lemma of the preceding section, $\lambda_k \to 0$ as $k \to \infty$, which contradicts the assumption that $|\lambda_k| > \rho > 0$ for every k.

COROLLARY 1: *Only the point zero* (0) *can be a limit point of the eigenvalues of a completely continuous operator in* R.

COROLLARY 2: *To each nonzero eigenvalue of a completely continuous operator in* R *belong only a finite number of linearly independent eigenvectors. In other words, each nonzero eigenvalue of a completely continuous operator in* R *has finite multiplicity.*

COROLLARY 3: *Each completely continuous operator in* R *has only countably many linearly independent eigenvectors belonging to nonzero eigenvalues.*

THEOREM 2: *If A is a completely continuous operator in* R *and if, for a fixed* $\lambda \neq 0$, *the equation*

$$Af - \lambda f = h \tag{1}$$

has a solution for each $h \in \mathrm{R}$, *then the equation*

$$Af - \lambda f = 0 \tag{2}$$

has the unique solution $f = 0$, *i.e.,* λ *is not an eigenvalue of the operator A.*

Proof: Supposing the contrary, we assume that equation (2) has a solution $f_0 \neq 0$. Thus,

$$Af_0 = \lambda f_0.$$

Solving equation (1) with $h = f_0$, we get a certain vector f_1:

$$Af_1 - \lambda f_1 = f_0.$$

Analogously, we find a vector f_2 such that

$$Af_2 - \lambda f_2 = f_1.$$

Continuing this process, we find an infinite sequence of vectors $\{f_k\}_1^\infty$ such that

$$Af_k - \lambda f_k = f_{k-1} \qquad (k = 1, 2, 3, \ldots).$$

We also have

$$Af_0 - \lambda f_0 = 0.$$

We show next that the vectors f_k $(k = 0, 1, 2, \ldots)$ are linearly independent. Supposing the contrary, we assume that in the sequence of vectors

$$f_0, f_1, f_2, \ldots, f_k, \ldots$$

the first one which is a linear combination of the preceding vectors is f_n, so that

$$f_n = \alpha_0 f_0 + \alpha_1 f_1 + \ldots + \alpha_{n-1} f_{n-1}. \tag{3}$$

Applying the operator A to both members of this equation, we obtain

$$\lambda f_n + f_{n-1} = \alpha_0 \lambda f_0 + \alpha_1 (\lambda f_1 + f_0) + \ldots + \alpha_{n-1} (\lambda f_{n-1} + f_{n-2}).$$

This equation and (3) yield

$$f_{n-1} = \alpha_1 f_0 + \alpha_2 f_1 + \ldots + \alpha_{n-1} f_{n-2},$$

which contradicts our assumption. Therefore, the vectors f_k $(k = 0, 1, 2, \ldots)$ are linearly independent. Orthogonalizing this sequence, we get the orthonormal sequence

$$\begin{aligned} g_0 &= \alpha_{00} f_0, \\ g_1 &= \alpha_{10} f_0 + \alpha_{11} f_1, \\ &\ldots\ldots\ldots \\ g_k &= \alpha_{k0} f_0 + \alpha_{k1} f_1 + \ldots + \alpha_{kk} f_k, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

It follows that

$$\begin{aligned} Ag_k &= \alpha_{k0}\lambda f_0 + \alpha_{k1}(\lambda f_1 + f_0) + \ldots + \alpha_{kk}(\lambda f_k + f_{k-1}) = \\ &= \alpha_{k1} f_0 + \alpha_{k2} f_1 + \ldots + \alpha_{kk} f_{k-1} + \lambda g_k = \\ &= \beta_{k0} g_0 + \beta_{k1} g_1 + \ldots + \beta_{k,k-1} g_{k-1} + \lambda g_k. \end{aligned}$$

Since this equation contradicts the lemma of the preceding section, our theorem is proved.

COROLLARY 4: *If for a fixed* $\lambda \neq 0$ *equation* (1) *is solvable for each* $h \in \mathrm{R}$, *then, given* $h \in \mathrm{R}$, *this equation has a unique solution and, consequently, the operator* $A - \lambda E$ *has an inverse operator* $(A - \lambda E)^{-1}$.

THEOREM 3: *Let* A *be a completely continuous operator in* R *and fix* $\lambda \neq 0$. *There exists a constant* L, *depending only on* A *and* λ, *such that if the equation*

$$Af - \lambda f = h, \tag{1}$$

with a given right member h, *has a solution* f, *then, for at least one of its solutions,*

$$\|f\| \leqq L\|h\|.$$

Proof: Fix h and assume that (1) has a solution f^*. If λ is an eigenvalue of A, let

$$f_1, f_2, \ldots, f_k$$

be a complete system of linearly independent eigenvectors of the operator A belonging to λ. In this case, the general solution of equation (1) has the form

$$f = f^* + \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k,$$

where $a_1, a_2, \ldots, a_k$ are arbitrary complex numbers. We select these numbers so that the norm $\|f\|$ of the vector f is a minimum. Let $\overset{0}{f}$ denote the solution of (1) with minimum norm. If λ is regular then $\overset{0}{f} = f^*$. Now let h run through the set M of all vectors for which the equation (1) is solvable. To each vector $h \in M$, there corresponds a minimal solution $\overset{0}{f}$. We must show that[1]

$$\sup_{h \in M} \frac{\|\overset{0}{f}\|}{\|h\|} < \infty.$$

We suppose the contrary. Then there exists a sequence of vectors $\{h_k\}_1^\infty$ such that, as $k \to \infty$

$$\frac{\|\overset{0}{f_k}\|}{\|h_k\|} \to \infty$$

where $\overset{0}{f_k}$ is the minimal solution of (1) with right member h_k. We divide both sides of the equation

$$A\overset{0}{f_k} - \lambda \overset{0}{f_k} = h_k \qquad (k = 1, 2, 3, \ldots)$$

by $\|\overset{0}{f_k}\|$ to get the equation

$$A\overset{0}{f'_k} - \lambda \overset{0}{f'_k} = h'_k \qquad (k = 1, 2, 3, \ldots)$$

where

$$h'_k = \frac{h_k}{\|\overset{0}{f_k}\|}, \|\overset{0}{f'_k}\| = 1 \qquad (k = 1, 2, 3, \ldots).$$

Thus, the minimal solution $\overset{0}{f'_k}$ of equation (1) has norm one if the right member is h'_k. Since the operator A is completely continuous, there exists a subsequence

$$\overset{0}{f'_{n_1}}, \overset{0}{f'_{n_2}}, \ldots$$

of $\{\overset{0}{f'_k}\}$ for which the limit

$$\lim_{i \to \infty} A\overset{0}{f'_{n_i}}$$

exists. Since

$$h'_k \to 0$$

as $k \to \infty$, the limit

$$\lim_{i \to \infty} A\overset{0}{f'_{n_i}} = g$$

---

[1] Translator's Note: In the displayed inequality $\overset{0}{f}$ is determined by h.

also exists and, consequently,

$$Ag - \lambda g = 0$$

where $\| g \| = 1$. Thus, g is an eigenvector of the operator A.

Both the vector $\overset{0}{f}'_{n_i} - g$, and the vector $\overset{0}{f}'_{n_i}$ are solutions of equation (1) with the right member h'_{n_i}. But, because the minimum norm of a solution of this equation is one, we have, for each i,

$$\| \overset{0}{f}'_{n_i} - g \| \geqq 1.$$

Since this is impossible, the theorem is proved.

53. Further Properties of Completely Continuous Operators

THEOREM 1: *If λ is a nonzero eigenvalue of the completely continuous operator A in the Hilbert space* H, *then $\bar{\lambda}$ is an eigenvalue of the operator A^*.*

Proof: Let the vector h run through H. Then the vector

$$g = Ah - \lambda h$$

does not run through the whole space, but only through some linear manifold G $\subset$ H, because the equation

$$Af - \lambda f = g$$

is not solvable for every right member g. It is not difficult to see that the manifold G is closed, so that G is a subspace. Indeed, if $g_n \in$ G ($n = 1, 2, 3, \ldots$) then, according to Theorem 3 of the preceding section, there exist a constant L and vectors $\overset{0}{h}_n$ for which

$$A\overset{0}{h}_n - \lambda \overset{0}{h}_n = g_n \qquad (n = 1, 2, 3, \ldots)$$

and

$$\| \overset{0}{h}_n \| \leqq L \| g_n \| \qquad (n = 1, 2, 3, \ldots).$$

If $g_n \to g$, then the sequence of vectors $\{\overset{0}{h}_n\}_{n=1}^{\infty}$ is bounded and, therefore, it has a subsequence $\{\overset{0}{h}_{n_i}\}_{i=1}^{\infty}$ such that

$$\lim_{i \to \infty} A\overset{0}{h}_{n_i}$$

exists. Consequently,

$$h = \lim_{i \to \infty} \overset{0}{h}_{n_i}$$

exists, and we obtain

$$Ah - \lambda h = g.$$

Hence, $g \in G$, so that the manifold G is closed. Since the subspace G does not coincide with H, there exists a nonzero vector f which is orthogonal to G. Then, for each $h \in H$,

$$(Ah - \lambda h, f) = 0$$

so that

$$(Ah, f) = (\lambda h, f)$$

or

$$(Ah, f) = (h, \bar{\lambda} f).$$

This relation shows that $f \in D_{A^*}$ and

$$A^* f = \bar{\lambda} f.$$

The theorem is proved.

Now we can strengthen Theorem 2 of the preceding section somewhat. This theorem asserts that if $\lambda \neq 0$ is an eigenvalue of the completely continuous operator A in R, then the equation

$$Af - \lambda f = g \tag{1}$$

is not solvable for every $g \in R$. Supposing that A is now defined in the Hilbert space H, we shall determine the set of vectors g for which equation (1) is solvable.

THEOREM 2: *Let A be a completely continuous operator in* H *and fix $\lambda \neq 0$. Then equation* (1) *has a solution if and only if the vector g is orthogonal to the eigenmanifold* F *of the operator A^* belonging to $\bar{\lambda}$. If $\bar{\lambda}$ is not an eigenvalue of the operator A^*, then* F *is understood to be the null subspace, i.e., in this case, equation* (1) *is solvable for every right member $g \in H$.*

Proof: Let G be the set of all vectors g having a representation

$$g = Ah - \lambda h,$$

for some $h \in H$. We must show that $H \ominus G$ coincides with the eigenmanifold F of the operator A^* belonging to $\bar{\lambda}$. Let the vector f be orthogonal to G. Then, repeating the corresponding part of the proof of Theorem 1, we find that

$$A^* f = \bar{\lambda} f.$$

Thus, $f \in F$, which implies that

$$H \ominus G \subset F.$$

In particular, if $\bar{\lambda}$ is not an eigenvalue of the operator A^*, then $H \ominus G = 0$, so that $G = H$ and equation (1) is solvable for every right member $g \in H$. It remains to show that if $\bar{\lambda}$ is an eigenvalue of the operator A^*, then

$$F \subset H \ominus G. \tag{2}$$

Let $f \in F$ and let g be any vector of the form

$$g = Ah - \lambda h.$$

Then

$$(f, g) = (f, Ah - \lambda h) = (f, Ah) - \bar{\lambda}(f, h) = (A^* f - \bar{\lambda} f, h) = 0.$$

This implies that $f \perp G$. Therefore, $F \perp G$ and relation (2) is proved. At the same time, the theorem is proved.

The reader familiar with the elementary theory of integral equations will recognize that the propositions established above are generalizations of two theorems of Fredholm.

A generalization of a third theorem of Fredholm is also valid. It is formulated as follows: *the eigenmanifolds of the completely continuous operators A and A^* belonging to the characteristic values λ and $\bar{\lambda}$ have the same dimension.* We shall not prove this theorem.

54. The Existence Theorem for Eigenvectors of Completely Continuous Self-Adjoint Operators

The fundamental theorem concerning completely continuous self-adjoint operators asserts: *every completely continuous self-adjoint nonzero operator A in R has at least one eigenvector e which belongs to a nonzero eigenvalue λ.*

We shall give two different proofs of this theorem.

First proof: Let

$$M = \sup_{\|g\| = 1} |(Ag, g)| = \sup_{\|g\| = 1} \|Ag\|.$$

Clearly, $M > 0$. It follows from the definition of the supremum that there exists a sequence of normalized vectors $\{g_n\}_1^\infty$ for which

$$\lim_{n \to \infty} (Ag_n, g_n)$$

exists, and is equal to $+M$ or $-M$. We denote this nonzero limit by λ. Since A is completely continuous, the bounded sequence $\{g_n\}_1^\infty$ has a subsequence $\{g_{n_i}\}_{i=1}^\infty$ for which

$$\lim_{i \to \infty} Ag_{n_i} = h \tag{1}$$

exists. From the equation

$$\|Ag_{n_i} - \lambda g_{n_i}\|^2 = \|Ag_{n_i}\|^2 - 2\lambda (Ag_{n_i}, g_{n_i}) + \lambda^2$$

it follows that

$$\lim_{i \to \infty} \|Ag_{n_i} - \lambda g_{n_i}\|^2 = \|h\|^2 - 2\lambda^2 + \lambda^2 = \|h\|^2 - \lambda^2. \tag{2}$$

Since

$$\| Ag_{n_i} \| \leqq M \| g_{n_i} \| = M = |\lambda|$$

equation (1) yields

$$\| h \| \leqq |\lambda| .$$

But, since the left member of equation (2) is non-negative, $\| h \| = |\lambda|$, so that (2) becomes

$$\lim_{i \to \infty} \| Ag_{n_i} - \lambda g_{n_i} \| = 0. \tag{3}$$

It follows from (1) and (3) that $\lim_{i \to \infty} g_{n_i}$ exists and is equal to $\frac{h}{\lambda}$. Introducing the vector $e = \frac{h}{\lambda}$, which has norm one, we obtain from (1) and (3) the equation

$$Ae - \lambda e = 0.$$

This completes the first proof.

Second proof: We choose an arbitrary vector f_0 for which $Af_0 \neq 0$. It is not difficult to prove that $A^n f_0 \neq 0$, $n = 1, 2, 3, \ldots$. Indeed, assume that $A^n f_0 = 0$ for some n. If n is even and $n = 2k$ then

$$0 = (A^n f_0, f_0) = (A^k f_0, A^k f_0).$$

If n is odd and $n = 2k - 1$ then

$$0 = (A^n f_0, Af_0) = (A^k f_0, A^k f_0).$$

Thus, for each n, if $A^n f_0 = 0$, then $A^k f_0 = 0$, where $k = \left[\frac{n}{2}\right]$. Repeating this argument, we get finally that $Af_0 = 0$, which is impossible.[2] From this result it follows that there exist two infinite sequences of nonzero vectors

$$\{f_k\}_0^\infty, \{f'_k\}_0^\infty$$

defined by means of the equations

$$f'_k = \frac{f_k}{\| f_k \|}, \quad f_{k+1} = Af'_k \qquad (k = 0, 1, 2, \ldots).$$

From these definitions it follows that

$$\| f_k \| \leqq \| f_{k+1} \| \qquad (k = 0, 1, 2, \ldots) \tag{4}$$

and

$$\| f_{k-1} \| \cdot \| f_k \| = (f_{k-1}, f_{k+1}) = (f_{k+1}, f_{k-1}) \qquad (k = 1, 2, 3, \ldots). \tag{5}$$

Indeed, for $k \geqq 1$,

$$\begin{aligned} \| f_k \| &= (f_k, f'_k) = (Af'_{k-1}, f'_k) = (f'_{k-1}, Af'_k) = \\ &= (f'_{k-1}, f_{k+1}) \leqq \| f'_{k-1} \| \cdot \| f_{k+1} \| = \| f_{k+1} \|, \end{aligned}$$

[2] Translator's Note: The proof is slightly easier if n is chosen as the *smallest* integer for which $A^n f_0 = 0$. Since $Af_0 \neq 0$ by hypothesis, $n > 1$. By the argument given above, $A^k f_0 = 0$ for some $k < n$. This is a contradiction.

which proves the inequality (4) and

$$(f'_{k-1}, f_{k+1}) = \|f_k\|,$$

which implies (5).

Again denote the norm of the operator A by M. Then

$$\|Af'_{k-1}\| \leqq M$$

and, consequently,

$$\|f_k\| \leqq M.$$

Therefore, $\{\|f_k\|\}_1^\infty$ is a bounded nondecreasing sequence of positive numbers, so that it has a finite (positive) limit:

$$\lim_{k\to\infty} \|f_k\| = \lambda. \tag{6}$$

Moreover, since A is completely continuous, there exists a subsequence $\{f'_{n_i}\}_{i=1}^\infty$ for which the sequence of vectors

$$f_{n_i+1} = Af'_{n_i} \qquad (i = 1, 2, 3, \ldots)$$

has a limit as $i \to \infty$:

$$f_{n_i+1} \to g.$$

Noting that

$$f_{n_i+2} = Af'_{n_i+1} = \frac{Af_{n_i+1}}{\|f_{n_i+1}\|},$$

we infer without difficulty that the sequence $\{f_{n_i+2}\}_{i=1}^\infty$ converges. Similarly, the convergence of the sequence $\{f_{n_i+3}\}_{i=1}^\infty$ is established. We suppose that, as $i \to \infty$,

$$f_{n_i+2} \to h, \; f_{n_i+3} \to h'$$

and evaluate $\|h' - g\|^2$. Using relations (5) and (6), we get

$$\|h' - g\|^2 = \lim_{i\to\infty} \|f_{n_i+3} - f_{n_i+1}\|^2 =$$

$$= \lim_{i\to\infty} \{\|f_{n_i+3}\|^2 + \|f_{n_i+1}\|^2 - (f_{n_i+3}, f_{n_i+1}) - (f_{n_i+1}, f_{n_i+3})\} = 0.$$

Therefore, $h' = g$. On the other hand, since

$$h = \frac{Ag}{\lambda}, \qquad h' = \frac{Ah}{\lambda}$$

we have

$$Ag = \lambda h, \qquad Ah = \lambda g,$$

so that

$$A(h+g) = \lambda(h+g), \quad A(h-g) = -\lambda(h-g).$$

The vector g is not equal to zero because $\|g\| = \lambda$. So at least one of the vectors $h + g$ and $h - g$ is different from zero. This nonzero vector is an eigenvector of the operator A belonging to an eigenvalue which is either λ or $-\lambda$.

55. The Spectrum of a Completely Continuous Self-Adjoint Operator in R

In the preceding section we proved that each nonzero completely continuous self-adjoint operator A in R has at least one nonzero eigenvalue λ. In this section we construct a complete system of nonzero eigenvalues of the operator A.

THEOREM: *Each nonzero completely continuous self-adjoint operator A has a finite or infinite orthonormal sequence of eigenvectors $e_1, e_2, e_3, \ldots$ belonging to its nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ ($|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$) such that for each vector f of the form $f = Ah$ the Parseval relation*

$$\|f\|^2 = \sum_k |(f, e_k)|^2$$

holds.

Proof: On the basis of the theorem of the preceding section there exists a vector e_1 such that $\|e_1\| = 1$ and

$$Ae_1 = \lambda_1 e_1$$

where

$$\lambda_1 = \pm \sup_{\|g\|=1} |(Ag, g)|.$$

For later convenience we re-denote our linear system R by R_1, and the operator A by A_1. Let

$$\mathrm{R}_2 = \mathrm{R}_1 \ominus e_1.$$

It is clear that R_2 is also a linear metrizable system. Moreover, if $f \in \mathrm{R}_2$, then $A_1 f \in \mathrm{R}_2$, since the equation $(f, e_1) = 0$ implies that

$$(A_1 f, e_1) = (f, A_1 e_1) = (f, \lambda_1 e_1) = \lambda_1 (f, e_1) = 0.$$

The restriction A_2 of the operator A_1 to R_2 is also a completely continuous self-adjoint operator. If the operator A_2 is not identically zero then, by the theorem of the preceding section, there exists a vector $e_2 \in \mathrm{R}_2$ such that

$$A_2 e_2 = \lambda_2 e_2 \qquad (\|e_2\| = 1).$$

Since $e_2 \in \mathrm{R}_2$ we have $(e_2, e_1) = 0$. Furthermore,

$$|\lambda_2| = \sup_{\|f\|=1, f \in \mathrm{R}_2} |(A_1 f, f)| \leq \sup_{\|g\|=1} |(A_1 g, g)| = |\lambda_1|.$$

In the same manner we define the linear system

$$\mathrm{R}_3 = \mathrm{R}_2 \ominus e_2$$

and let A_3 denote the restriction of A_1 to R_3. If $A_3 \neq 0$, we select a normalized eigenvector e_3 and the corresponding eigenvalue λ_3. This process comes to an end at the nth step only if the restriction A_n of the operator

A_1 to R_n turns out to be identically zero. In that case we get a finite orthonormal sequence of vectors

$$e_1, e_2, \ldots, e_{n-1},$$

belonging to the nonzero eigenvalues

$$\lambda_1, \lambda_2, \ldots \lambda_{n-1},$$

where

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_{n-1}|$$

and

$$|\lambda_k| = \sup_{\|f\|=1, f \in R_k} |(A_1 f, f)|.$$

If the process does not come to an end after a finite number of steps, then we get an infinite orthonormal sequence $\{e_k\}_1^\infty$ such that the corresponding sequence of eigenvalues converges to zero: $\lambda_k \to 0$ as $k \to \infty$. Let m be the numbers of elements of the sequence $\{e_k\}$ if the sequence is finite, and let m be an arbitrary fixed integer otherwise. Now we choose $h \in H$ arbitrarily and let

$$g = h - \sum_{k=1}^{m} (h, e_k)\, e_k.$$

Since

$$(g, e_k) = 0 \qquad (k = 1, 2, 3, \ldots, m)$$

we have $g \in R_{m+1}$. Therefore,

$$\|Ag\|^2 \leqq \|A_{m+1}\|^2_{R_{m+1}} \cdot \|g\|^2$$

or

$$\|Ah - \sum_{k=1}^{m} (h, e_k)\, Ae_k\|^2 \leqq \|A_{m+1}\|^2_{R_{m+1}} \cdot \|g\|^2. \tag{1}$$

Noting that

$$(h, e_k)\, Ae_k = (h, e_k)\, \lambda_k e_k = (h, \lambda_k e_k) e_k = (h, Ae_k) e_k = (Ah, e_k) e_k$$

and also that

$$\|g\| \leqq \|h\|$$

and letting $f = Ah$, we obtain from (1) the inequality

$$\|f - \sum_{k=1}^{m} (f, e_k)\, e_k\|^2 \leqq \|A_{m+1}\|^2_{R_{m+1}} \cdot \|h\|^2.$$

If the sequence

$$e_1, e_2, e_3, \ldots$$

is finite, then $A_{m+1} = 0$, so that (2) yields

$$f = \sum_{k=1}^{m} (f, e_k)\, e_k.$$

If the sequence $\{e_k\}$ is infinite, then it follows from (2) that

$$\|f-\sum_{k=1}^{m}(f,e_k)\,e_k\|^2 \leqq \lambda_{m+1}^2\,\|h\|^2$$

which yields

$$\|f\|^2-\sum_{k=1}^{m}|(f,e_k)|^2 \leqq \lambda_{m+1}^2\,\|h\|^2.$$

Letting $m\to\infty$, we get

$$\|f\|^2=\sum_{k=1}^{\infty}|(f,e_k)|^2.$$

56. Completely Continuous Normal Operators

Let S denote a completely continuous normal operator in the linear system R. We shall consider the operator $A=S^*S=SS^*$, which is completely continuous, self-adjoint, and nonnegative:

$$(Af,f)\geqq 0$$

for every $f\in$ R. The nonnegativity of A follows from the fact that

$$(Af,f)=(S^*Sf,f)=(Sf,Sf).$$

This property implies that all the eigenvalues of A are nonnegative. We denote them by

$$\rho_1^2\geqq\rho_2^2\geqq\rho_3^2\geqq\ldots.$$

On the basis of the preceding section the operator A has a complete orthonormal system of eigenvectors g_k:

$$Ag_k=\rho_k^2 g_k \qquad (k=1,2,3,\ldots).$$

We choose a positive eigenvalue of the operator A (let it be ρ^2) and suppose that its multiplicity is r. Let

$$g^{(1)},g^{(2)},\ldots,g^{(r)},$$

(1) $$(g^{(i)},g^{(k)})=\delta_{ik} \qquad (i,k=1,2,3,\ldots,r),$$

comprise a complete orthonormal system of eigenvectors of the operator A belonging to the eigenvalue ρ^2. We define $h^{(i)}$ such that

$$S^*g^{(i)}=\rho h^{(i)} \qquad (i=1,2,3,\ldots,r).$$

Then

$$\rho Sh^{(i)}=SS^*g^{(i)}=Ag^{(i)}=\rho^2 g^{(i)}$$

so that

$$Sh^{(i)}=\rho g^{(i)} \qquad (i=1,2,3,\ldots,r).$$

Since

$$(h^{(i)},h^{(k)})=\frac{1}{\rho^2}(S^*g^{(i)},S^*g^{(k)})=\frac{1}{\rho^2}(Ag^{(i)},g^{(k)})=(g^{(i)},g^{(k)})$$

the vectors $h^{(i)}$ comprise an orthonormal system:

(2) $$(h^{(i)},h^{(k)})=\delta_{ik} \qquad (i,k=1,2,3,\ldots,r).$$

Furthermore,

$$Ah^{(i)} = \frac{1}{\rho} SS^*S^*g^{(i)} = \frac{1}{\rho} S^*Ag^{(i)} = \rho S^*g^{(i)} = \rho^2 h^{(i)} \qquad (i = 1, 2, 3, \ldots, r).$$

So $h^{(i)}$ is an eigenvector of the operator A which belongs to the eigenvalue ρ^2. Hence, $h^{(i)}$ is a linear combination of the vectors

$$g^{(1)}, g^{(2)}, \ldots, g^{(r)},$$

i.e.,[3]

$$(3) \qquad h^{(i)} = a_{i1}g^{(1)} + a_{i2}g^{(2)} + \ldots + a_{ir}g^{(r)}.$$

We consider the n-dimensional space G spanned by the vectors $g^{(i)}$ ($i = 1, 2, 3, \ldots, r$). Let U denote the linear operator in G which is defined by

$$Ug^{(i)} = h^{(i)} \qquad (i = 1, 2, 3, \ldots, r).$$

Relations (1) and (2) imply that U is a unitary operator. We proceed to determine the eigenvectors of U. If f is an eigenvector and

$$f = x_1 g^{(1)} + x_2 g^{(2)} + \ldots + x_r g^{(r)},$$

then from the equation

$$Uf = \theta f$$

it follows that

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + \ldots + a_{r1}x_r &= \theta x_1, \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{r2}x_r &= \theta x_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{1r}x_1 + a_{2r}x_2 + \ldots + a_{rr}x_r &= \theta x_r . \end{aligned}$$

Thus, we obtain the equation

$$\begin{vmatrix} a_{11} - \theta & a_{21} & \ldots & a_{r1} \\ a_{12} & a_{22} - \theta & \ldots & a_{r2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1r} & a_{2r} & \ldots & a_{rr} - \theta \end{vmatrix} = 0.$$

Conversely, each root θ of this equation corresponds to an eigenvector of the operator U. Since U is unitary, each root has absolute value one. Let $\theta^{(1)}$ be one of these roots and let $f^{(1)}$ be the corresponding eigenvector. Consider the subspace G_2 consisting of all vectors of the space $G \equiv G_1$ which are orthogonal to $f^{(1)}$. Obviously, U is a unitary operator in G_2. Therefore, repeating the foregoing argument, we find that the operator U has an eigenvector $f^{(2)}$ in G_2 with corresponding eigenvalue $\theta^{(2)}$. Continuing this "splitting off" process, we construct an orthonormal system of r vectors

$$f^{(1)}, f^{(2)}, \ldots, f^{(r)}.$$

---

[3] Translator's Note: Since the $g^{(i)}$ are linearly independent, the coefficients in (3) are unique.

These vectors satisfy the equations

$$Uf^{(i)} = \theta^{(i)} f^{(i)} \qquad (i = 1, 2, 3, \ldots, r).$$

The vectors $f^{(i)}$ also span the space G. If $f^{(i)}$ is given in terms of the vectors $g^{(i)}$ by the equation

$$f^{(i)} = x_1^{(i)} g^{(1)} + x_2^{(i)} g^{(2)} + \ldots + x_r^{(i)} g^{(r)}$$

then

$$S^* f^{(i)} = \rho \{ x_1^{(i)} h^{(1)} + x_2^{(i)} h^{(2)} + \ldots + x_r^{(i)} h^{(r)} \} = \rho\, U f^{(i)} = \rho\, \theta^{(i)} f^{(i)}.$$

Therefore, $f^{(i)}$ is an eigenvector of the operator S^* belonging to the eigenvalue $\rho\theta^{(i)}$. Analogously it is proved that $f^{(i)}$ is an eigenvector of the operator S belonging to the eigenvalue $\rho\bar{\theta}^{(i)}$. In the system of eigenvectors $g_1, g_2, g_3, \ldots$ of the operator A we now replace

$$g^{(1)}, g^{(2)}, \ldots, g^{(r)}$$

by the vectors

$$f^{(1)}, f^{(2)}, \ldots, f^{(r)}.$$

If we proceed in the same way with each eigenvalue of the operator A having multiplicity larger than 1, we get the proof of the first part of the following theorem.

THEOREM: *If S is a completely continuous nonzero normal operator in* R, *then there exists an orthonormal system of vectors $\{e_k\}$ and a system of nonzero (complex) numbers $\{\lambda_k\}$ for which*

$$Se_k = \lambda_k e_k, \qquad S^* e_k = \bar{\lambda}_k e_k \qquad (k = 1, 2, 3, \ldots).$$

This system of vectors is complete in the sense that each element f of the form Sh or S^*h is represented by the series

$$f = \sum_k (f, e_k) e_k. \tag{4}$$

In order to prove the second part of the assertion we shall let $f = Sh$ and consider the vector

$$f' = S^* f = S^* Sh = Ah.$$

The theorem in Section 55 on completely continuous self-adjoint operators and the fact[4] that the vectors e_k form an orthonormal system of eigenvectors of A which is complete in the sense of this theorem yield the Parseval relation,

$$\| f' \|^2 = \sum_{k=1}^{\infty} | (f', e_k) |^2.$$

Thus, f' is the strong limit of the sequence

$$\sum_{k=1}^{n} (f', e_k)\, e_k \qquad (n = 1, 2, 3, \ldots).$$

[4] We assume for definiteness that they form an infinite set.

Therefore,

$$(f', h) = \lim_{n \to \infty} \sum_{k=1}^{n} (f', e_k)(e_k, h) = \sum_{k=1}^{\infty} (f', e_k)(e_k, h).$$

But since

$$(f', h) = (S^* f, h) = (f, Sh) = (f, f),$$
$$(f', e_k) = (S^* f, e_k) = (f, Se_k) = \bar{\lambda}_k (f, e_k),$$
$$(e_k, h) = \frac{1}{\bar{\lambda}_k} (S^* e_k, h) = \frac{1}{\bar{\lambda}_k} (e_k, Sh) = \frac{1}{\bar{\lambda}_k} (e_k, f)$$

the Parseval relation yields

$$\|f\|^2 = \sum_{k=1}^{\infty} |(f, e_k)|^2.$$

Therefore, equation (4) is proved if $f = Sh$. A similar proof works if $f = S^* h$.

57. Applications to the Theory of Almost Periodic Functions

As was indicated in Section 13, a (complex) continuous function $f(t)$ $(-\infty < t < \infty)$ is *almost periodic* if for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that each real interval of length $l(\varepsilon)$ contains at least one number τ for which

$$|f(t + \tau) - f(t)| < \varepsilon$$

for all t. Each such τ is called a *translation number*.

Starting from this definition, it is not difficult to establish several simple properties of almost periodic functions. We list these properties without proof.[5]

I. Every almost periodic function is bounded and uniformly continuous on the whole real axis.

II. For each almost periodic function $f(t)$ there exists the so-called *mean value*

$$\mathbf{M}\{f(t)\} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+\alpha}^{T+\alpha} f(t)\, dt.$$

The right hand limit is uniform in α.

III. The product and sum of two almost periodic functions are almost periodic.

IV. If $f(t)$ is an almost periodic function and if

$$\mathbf{M}\{|f(t)|^2\} = 0,$$

then $f(t) \equiv 0$.

[5] The reader will find a detailed presentation in the survey article by B. M. Levitan [1].

The set of all almost periodic functions becomes a metrized linear system R if the scalar product is defined by the formula

$$(f,g) = \mathbf{M}\{f(t)\overline{g(t)}\}. \tag{1}$$

Now property IV has the equivalent form: $(f,f) = 0$ implies $f = 0$.

In Section 13, the scalar product (1) was introduced for all polynomials of the form

$$\sum_k A_k e^{i\lambda_k t}. \tag{2}$$

Being the sum of periodic functions which are, therefore, almost periodic, every such polynomial belongs to our linear system R. In particular, the linear system R contains the continuum of functions, $e^{i\lambda t}$ $(-\infty < \lambda < \infty)$, which form an orthonormal system. Therefore, as was the case with the space B^2 defined in Section 13, the linear system R is nonseparable.

The central place in the theory of almost periodic functions is occupied by the theorem, mentioned in Section 13, to the effect that each almost periodic function is the uniform limit on the whole real axis of polynomials of the form (2). As was first shown by Bohr, it is possible to prove this theorem with the aid of a particular method of harmonic analysis of almost periodic functions. Instead of the usual Fourier coefficients of pure periodic functions, we introduce in the harmonic analysis of the almost periodic function $f(t)$ the function

$$a(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}\,dt = \mathbf{M}\{f(t)e^{-i\lambda t}\} = (f, e^{i\lambda t}).$$

The existence of $a(\lambda)$ for each real λ follows from properties II and III. On the basis of the remark preceding Theorem 4 in Section 9, for each function $f(t) \in \mathrm{R}$ there exist only countably many values of the parameter λ for which $a(\lambda) \neq 0$. We denote these values by $\lambda_1, \lambda_2, \ldots$ and we call them the *Fourier exponents* of the function $f(t)$. We call the corresponding numbers

$$C_k = a(\lambda_k) \qquad (k = 1, 2, 3, \ldots)$$

the *Fourier coefficients* of the function $f(t)$. Thus, to each function $f(t) \in \mathrm{R}$ there corresponds a *Fourier series:*

$$f(t) \sim \sum_k C_k e^{i\lambda_k t}.$$

From the general situation of Section 8, it follows that

$$\sum_k |C_k|^2 \leqq \mathbf{M}\{|f(t)|^2\}. \tag{3}$$

This is a generalization of the so-called *Bessel's inequality.*

The fundamental theorem of the Bohr theory asserts that there is equality in (3) for each almost periodic function $f(t)$. We present next a proof of the Bohr theorem. The proof, which is due to H. Weyl, is based on the theory of completely continuous operators. For this purpose, fix $f(t) \in \mathbf{R}$ and let

$$v(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s-t)\,u(t)\,dt = \underset{t}{\mathbf{M}}\{f(s-t)\,u(t)\}.$$

This formula associates with each function $u(t) \in \mathbf{R}$ a function $v(t)$ which, it is evident, also belongs to R. Thus, we have a linear operator, determined by the function $f(t)$, which maps $u(t)$ into $v(t)$. We denote this operator by A.

We shall prove that A is a normal operator. For this purpose we introduce a second operator B: let $w = Bu$ if

$$w(s) = \underset{t}{\mathbf{M}}\{\overline{f(t-s)}\,u(t)\}.$$

We shall prove that

(a) $(Au_1, u_2) = (u_1, Bu_2)$ for $u_1(t), u_2(t) \in \mathbf{R}$,
(b) $AB = BA$.

As for property (a), we have

$$\begin{aligned}(Au_1, u_2) &= \underset{s}{\mathbf{M}}\{\underset{t}{\mathbf{M}}[f(s-t)\,u_1(t)]\overline{u_2(s)}\} = \\ &= \underset{t}{\mathbf{M}}\{u_1(t)\underset{s}{\mathbf{M}}[f(s-t)\overline{u_2(s)}]\} = \\ &= \underset{t}{\mathbf{M}}\{u_1(t)\underset{s}{\mathbf{M}}[\overline{f(s-t)}\,u_2(s)]\} = (u_1, Bu_2).\end{aligned}$$

The reversal of the order of the operations $\underset{s}{\mathbf{M}}$ and $\underset{t}{\mathbf{M}}$ is valid because the mean value is a uniform limit according to the alternate definition given in property II above. We now prove property (b). First, we note that

$$\begin{aligned}ABu &= \underset{t}{\mathbf{M}}\{f(s-t)\underset{\tau}{\mathbf{M}}[\overline{f(\tau-t)}\,u(\tau)]\} = \\ &= \underset{\tau}{\mathbf{M}}\{u(\tau)\underset{t}{\mathbf{M}}[f(s-t)\overline{f(\tau-t)}]\}.\end{aligned}$$

Next, we consider in detail the function of s and τ defined by

$$\varphi = \underset{t}{\mathbf{M}}[f(s-t)\overline{f(\tau-t)}] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s-t)\overline{f(\tau-t)}\,dt.$$

After the change of varable from t to s indicated by

$$\tau - t = \sigma - s$$

we get

$$\varphi = \lim_{T\to\infty} \frac{1}{2T} \int_{\tau+s-T}^{\tau+s+T} f(\sigma-\tau)\overline{f(\sigma-s)}\,d\sigma =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma-\tau)\overline{f(\sigma-s)}\,d\sigma = \underset{\sigma}{M}\,[f(\sigma-\tau)\overline{f(\sigma-s)}].$$

Hence,

$$ABu = \underset{\tau}{\mathbf{M}}\,\{u(\tau)\,\underset{\sigma}{\mathbf{M}}\,[f(\sigma-\tau)\overline{f(\sigma-s)}\,]\} =$$

$$= \underset{\sigma}{\mathbf{M}}\,\{\overline{f(\sigma-s)}\,\underset{\tau}{\mathbf{M}}\,[f(\sigma-\tau)\,u(\tau)]\} = BAu,$$

and property (*b*) is proved.

Now we are ready to prove that A is a completely continuous operator. Let G denote an infinite set of functions $u(t) \in \mathrm{R}$ for which

$$\|u\|^2 = \mathbf{M}\{|u(t)|^2\} \leqq 1.$$

We must prove that G contains a sequence $\{u_n(t)\}_1^\infty$ such that the corresponding sequence $\{v_n(t)\}_1^\infty$, where $v_n = Au_n$, converges in the norm of R to some function $V(t) \in \mathrm{R}$. The function $u(t) \in \mathrm{G}$ is transformed by the operator A into the function

$$v(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s-t)\,u(t)\,dt \tag{4}$$

so that

$$|v(s)| \leqq \lim_{T\to\infty} \sqrt{\frac{1}{2T}\int_{-T}^{T} |f(s-t)|^2\,dt}\ \sqrt{\frac{1}{2T}\int_{-T}^{T} |u(t)|^2\,dt}$$

or

$$|v(s)| \leqq \sqrt{\mathbf{M}\{|f(t)|^2\}} = C.$$

Thus, the transformed function $v(s)$ is uniformly bounded. Furthermore, from formula (4) it follows that

$$v(s') - v(s'') = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{f(s'-t) - f(s''-t)\}\,u(t)\,dt$$

which yields

$$|v(s') - v(s'')| \leqq \sqrt{\underset{t}{\mathbf{M}}\{|f(s'-t) - f(s''-t)|^2\}}.$$

This inequality shows that the function $v(s)$ is uniformly continuous. Indeed, $f(t)$ is uniformly continuous, so that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(s' - t) - f(s'' - t)| \leqq \varepsilon \qquad (-\infty < t < \infty)$$

for $|s' - s''| \leqq \delta$. Hence, for $|s' - s''| \leqq \delta$,

$$|v(s') - v(s'')| \leqq \sqrt{\mathbf{M}\{\varepsilon^2\}} = \varepsilon.$$

According to the well-known theorem of Arzelà the set of functions $v(t)$ contains a sequence $\{v_n(t)\}_1^\infty$ which is uniformly convergent on the whole real axis. We denote the limit function by $V(t)$. It also satisfies the inequality

$$|V(s') - V(s'')| \leqq \sqrt{\mathbf{M}_t\{|f(s' - t) - f(s'' - t)|^2\}}.$$

Therefore, if τ is a translation number of the function $f(t)$ corresponding to ε and if $s' - s'' = \tau$, then

$$|V(s') - V(s'')| \leqq \varepsilon.$$

From this it is evident that $V(t)$ is an almost periodic function. Since

$$\sqrt{\mathbf{M}\{|w_1(t) - w_2(t)|^2\}} \leqq \sup_{-\infty < t < \infty} |w_1(t) - w_2(t)|$$

the sequence $\{v_n(t)\}_{n=1}^\infty$ converges not only uniformly on the real axis, but also in the norm of R. It follows that the operator A is completely continuous.

Let μ be a nonzero eigenvalue of A and let

$$g_1(t), g_2(t), \ldots, g_n(t) \tag{5}$$

be a complete orthonormal system of eigenfunctions (in other words, eigenvectors) of the operator A belonging to the eigenvalue μ. If $g(t)$ is any one of these eigenfunctions, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s - t) g(t)\, dt = \mu g(s).$$

It follows that for each real number σ the function $g(t + \sigma)$ also is an eigenfunction belonging to μ. From this fact we obtain the equations

$$g_r(t + \sigma) = \sum_{k=1}^{n} c_{rk}(\sigma) g_k(t) \qquad (r = 1, 2, 3, \ldots, n). \tag{6}$$

Thus, the almost periodic functions in the orthonormal system (5) satisfy the functional equations (6). By various methods it is possible to prove that each of the functions $g_r(t)$ has the form[6]

$$g_r(t) = \sum_{k=1}^{N} c_r^{(k)} e^{i\lambda_k t}$$

where the $c_r^{(k)}$ are constants. If necessary, one can replace the original eigenfunctions $g_r(t)$ by linear combinations of them to obtain the eigenfunctions

$$e^{i\lambda_1 t}, e^{i\lambda_2 t}, \ldots, e^{i\lambda_n t}.$$

According to the definition of the function $\alpha(\lambda)$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s-t)\, e^{i\lambda_r t}\, dt = e^{i\lambda_r s} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s-t)\, e^{i\lambda_r (t-s)}\, dt = e^{i\lambda_r s}\, \alpha(\lambda_r).$$

Thus, we have found the general form of the eigenfunctions of the operator A and have proved that the eigenfunction $e^{i\lambda_r t}$ corresponds to the eigenvalue which is the Fourier constant

$$\alpha(\lambda_r) = \mathbf{M}\{f(t) e^{-i\lambda_r (t)}\}.$$

Let

$$e^{i\lambda_1 t}, e^{i\lambda_2 t}, e^{i\lambda_3 t}, \ldots$$

be the sequence (in some order) of all eigenfunctions of the operator A and let

$$C_1 = \alpha(\lambda_1),\ C_2 = \alpha(\lambda_2),\ C_3 = \alpha(\lambda_3), \ldots$$

be the corresponding eigenvalues. We consider the series

$$\sum_{k=1}^{\infty} C_k (\Phi, e^{i\lambda_k t})\, e^{i\lambda_k t}, \tag{7}$$

where $\Phi \in \mathbf{R}$. Since

$$\sum_{k=m}^{\infty} |C_k (\Phi, e^{i\lambda_k t})| \leqq \sqrt{\sum_{k=m}^{\infty} |C_k|^2}\ \sqrt{\sum_{k=m}^{\infty} |(\Phi, e^{i\lambda_k t})|^2} \leqq$$

$$\leqq \sqrt{\sum_{k=m}^{\infty} |C_k|^2}\ \sqrt{\mathbf{M}\{|\Phi(t)|^2\}}$$

[6] For this one must first show that the functions $c_{rk}(\sigma)$ are continuously differentiable. Then the system (6) yields the system of differential equations,

$$g_r'(t) = \sum_{k=1}^{n} c_{rk}'(0)\, g_k(t) \qquad (r = 1, 2, 3, \ldots, n).$$

The solutions of these equations are linear combinations of functions of the form

$$(\alpha_0 + \alpha_1 t + \ldots + \alpha_m t^m)\, e^{\beta t}.$$

Since each function $g_k(t)$ is bounded uniformly, β must be pure imaginary and the polynomial factor must reduce to the constant term.

Another proof, which is richer in ideas, is based on the simultaneous reduction to diagonal form of the Abelian group of unitary matrices

$$(c_{rk}(\sigma))_{r,\,k=1}$$

the series (7) converges uniformly in t. Now we examine the function

$$\mathop{\mathbf{M}}_{s}\{f(t-s)\overline{f(-s)}\}.$$

It is of the form Ag where $g(t) \in \mathrm{R}$. According to the theorem in Section 56, this function can be represented by the series

$$\sum_{k=1}^{\infty} C_k \mathop{\mathbf{M}}_{s}\{\overline{f(-s)}\, e^{-i\lambda_k s}\}\, e^{i\lambda_k t}$$

which converges in the norm of R. By what was proved above[7] this series converges uniformly, so that

$$\mathop{\mathbf{M}}_{s}\{f(t-s)\overline{f(-s)}\} = \sum_{k=1}^{\infty} C_k \mathop{\mathbf{M}}_{s}\{\overline{f(-s)}\, e^{-i\lambda_k s}\}\, e^{i\lambda_k t}.$$

Hence, for $t = 0$,

$$\mathbf{M}\{|f(-s)|^2\} = \sum_{k=1}^{\infty} C_k \mathop{\mathbf{M}}_{s}\{\overline{f(-s)}\, e^{-i\lambda_k s}\}$$

or

$$\mathbf{M}\{|f(t)|^2\} = \sum_{k=1}^{\infty} |C_k|^2,$$

and, finally, the fundamental theorem of Bohr is proved.

[7] The role of the function $\Phi(t)$ is now played by $f(\overline{-t})$.

BIBLIOGRAPHY

AKHIEZER, N. I. — [1] On some inversion formulas of singular integrals, *Izv. Akad. Nauk SSSR*, **9** (1945), 275–290; [2] Infinite Jacobi matrices and the moment problem, *Uspekhi mat. Nauk*, **9** (1941); [3] Integral operators with Carleman kernels, *Uspekhi mat. Nauk II*, **5** (21) (1947); [4] On the theory of maximal symmetric operators in Hilbert space, *Učenie Zapiski Av. Inst III*, **2** (1940).

AMBROSE, W. — [1] Spectral resolution of groups of unitary operators, *Duke Math. Journal*, **11** (1944), 589–595.

ARONZAJN, N. — [1] Caractérisation métrique de l'espace de Hilbert, etc., *C. R. Acad. Sci. Paris*, **201** (1935), 811–813, 873–875.

BANACH, S. — [1] *Théorie des opérations linéaires*, Warsaw (1932); [2] Sur les fonctionnelles linéaires, *Studia Math.*, **1** (1929), 211–216.

BEURLING, A. — [1] Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle, *IX Congrès des Math. Scandinaves, Helsingfors* (1938), 345–366.

BOCHNER, S. — [1] *Vorlesungen über Fouriersche Integrale*, Leipzig, (1932); [2] Spektraldarstellung linearer Scharen unitärer Operatoren, *Sitzber. Preuss. Akad. Wiss.* (1933), 371–376; [3] Inversion formulae and unitary transformations, *Ann. of Math.*, **35** (1934), 111–115.

BOHNENBLUST, H. F. and A. SOBCZYK — [1] Extensions of functionals on complex linear spaces, *Bulletin Amer. Math. Soc.*, **44** (1938), 91–93.

CARATHEODORY, C. — [1] Bemerkungen zum Riesz-Fischerschen Satz und zur Ergodentheorie, *Abhandl. Math. Sem. d. Hansischen Univ.*, **14** (1941), 351–398.

CARLEMAN, T. — [1] *Sur les équations intégrales singulières à noyau réel symétrique*, Uppsala (1923); [2] Zur Theorie der linearen Integralgleichungen, *Math. Zeitschr.*, **9** (1921), 196–217; [3] Application de la théorie des équations intégrales linéaires aux équations différentielles non linéaires, *Acta Math.*, **59** (1932), 63–87.

COOPER, J. L. B. — [1] The spectral analysis of self-adjoint operators, *Quarterly Jour. of Math.*, **16** (1945), 31–48; [2] One-parameter semigroups of isometric operators in Hilbert space, *Ann. of Math.*, **48** (1947), 827–842; [3] Symmetric operators in Hilbert space, *Proc. London Math. Soc.*, **50** (1948), 11–55.

COURANT, R. — [1] Zur Theorie der linearen Integralgleichungen, *Math. Ann.*, **89** (1923), 161–178.

COURANT, R. and D. HILBERT — [1] *Methoden der mathematischen Physik*, Berlin (1931).

DELSARTE, J. — [1] *Les groupes de transformations linéaires dans l'espace de Hilbert*, Mém. Sci. math., **57** (1923).

DOETSCH, G. — [1] *Theorie und Anwendung der Laplace-Transformation*, Berlin (1937).

DUNFORD, N. — [1] Spectral theory I, Convergence to projections, *Trans. Amer. Math. Soc.*, **54** (1943), 185–217; [2] Spectral theory, *Bulletin Amer. Math. Soc.*, **49** (1943), 637–651.

EBERLEIN, W. F. — [1] A note on the spectral theorem, *Bulletin Amer. Math. Soc.*, **52** (1946), 328–331.

EVANS, G. C. — [1] *Functionals and their applications.* Selected topics, including integral equations, Amer. Math. Soc. Coll. Publ. V, New York (1918).

FRECHET, M. — [1] Sur les opérations linéaires, *Trans. Amer. Math. Soc.*, **8** (1907), 433–446.

FREDHOLM, J. — [1] Les équations intégrales linéaires, *C. R. du Congr. de Stockholm*, (1910), 92–100.

FREUDENTHAL, H. — [1] Über die Friedrichssche Fortsetzung halbbeschränkter Hermitescher Operatoren, *Proc. Acad. Amsterdam*, **39** (1936), 832–833.

FRIEDRICHS, K. — [1] Spektraltheorie halbbeschränkter Operatoren, *Math. Ann.*, **109** (1934), 465–487; 685–713; also **110** (1935), 777–779; [2] Über die ausgezeichnete Randbedingung in der Spektraltheorie der halbbeschränkten gewöhnlichen Differentialoperatoren zweiter Ordnung, *Math. Ann.*, **112** (1935), 1–23; [3] Beiträge zur Theorie der Spektralschar, *Math. Ann.*, **110** (1935), 54–62; [4] Die unitären Invarianten selbstadjungierter Operatoren im Hilbertschen Raum, *Jber. Deutsch. Math.-Vereinig.*, **45** (1935), 79–82; [5] On differential operators in Hilbert spaces, *Amer. J. Math.*, **61** (1939), 523–544.

FUKAMIYA, M. — [1] On one-parameter groups of operators, *Proc. Imp. Acad. Sci. Tokyo*, **16** (1940), 262–265.

GELFAND, I. M. — [1] Sur une lemme de la théorie des espaces linéaires, *Soobšč. Chark. mat. Obšč.* (4), **13** (1936), 35–40; [2] Abstract functions and linear operators, *Mat. Sbornik N. S.*, **4** (1938), 235–284.

GLAZMAN, I. M. — [1] On the theory of singular quasi-differential operators, *Chark.*, (1949); [2] A class of solutions of a classical moment problem, *Zapiski naučno-issl. Inst. mat. i mech. i Chark. mat. Obšč.* (4), **20** (1950); [3] On the theory of singular differential operators, *Uspekhi mat. Nauk*, **5** (part 6) (1950).

GLAZMAN, I. — [1] On the defect index of differential operators, *Doklady Akad. Nauk SSSR*, **64** (#2) (1949).

GODEMENT, R. — [1] Sur une généralisation d'un théorème de Stone, *Comptes Rendus Acad. Sci. Paris*, **218** (1944), 901–903.

GRAFF, A. A. — [1] On the theory of linear differential systems in one-dimensional regions, *Mat. Sbornik*, **60** (1946); also **63** (1947).

HAHN, H. — [1] Über Folgen linearer Operationen, *Monatshefte Math. Phys.*, **32** (1922), 3–88.

HALMOS, P. R. — [1] *Introduction to Hilbert space and the theory of spectral multiplicity*, New York (1951).

HALPERIN, I. — [1] Closures and adjoints of linear differential operators, *Ann. of Math.*, **38** (1937), 880–919.

HAMBURGER, H. — [1] Beiträge zur Konvergenztheorie der Stieltjesschen Kettenbrüche, *Math. Zeitschr.*, **4** (1919), 186–222.

HELLINGER, E. and O. TOEPLITZ — [1] Grundlagen für eine Theorie der unendlichen Matrizen, *Math. Ann.*, **69** (1910), 289–330; [2] Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten, *Enzyklopädie d. Math. Wiss. II C*, **13**, Leipzig (1928).

HILBERT, D. — [1] *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig (1912); [2] Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variablen, *Rend. Circ. Math. Palermo*, **27** (1909), 59–74.

HILDEBRAND, T. H. — [1] Über vollstetige, lineare Transformationen, *Acta Math.*, **51** (1928), 311–318.

HILLE, E. — [1] *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publ., New York (1948); [2] Notes on linear transformations, *Trans. Amer. Math. Soc.*, **39** (1936), 131–153; [3] On semi-groups of transformations in Hilbert space, *Proc. Nat. Acad. Sci.*, **24** (1938) 159–161; [4] Analytical semi-groups in the theory of linear transformations, *Neuvième Congrès Math. Scand., Helsingfors* (1938), 135–145.

HILLE, E. and J. D. TAMARKIN — [1] On the theory of linear integral equations I, *Ann. of Math.* (2), **31** (1930), 479–528.

JULIA, G. — [1] *Introduction mathématique aux théories quantiques II*, Paris (1938).

KAKUTANI, S. — [1] Iteration of linear operators in complex Banach spaces, *Proc. Imp. Acad. Tokyo*, **14** (1938), 295–300.

KAZ, I. — [1] On Hilbert spaces generated by monotone Hermitean matrix functions, *Zapiski naučno-issl. Inst. mat. i mech. i Chark. mat. Obšč.*, **22** (1950).

KODAIRA, K. — [1] On some fundamental theorems in the theory of operators in Hilbert space, *Proc. Imp. Acad. Tokyo*, **15** (1939), 207–210.

KRASNOSELSKI, M. A. — [1] On defect numbers of closed operators, *Doklady Akad. Nauk SSSR*, **56** (1947), 559–561; [2] On self-adjoint extensions of Hermitean operators, *Ukrainskij mat. J. A. N. USSR*, **1** (1949), 21–38.

KREIN, M. — [1] On resolvents of Hermitean operators with defect indices (m, m), *Doklady Akad. Nauk SSSR*, **52** (#8) (1946), 657–660; [2] On a general method of decomposition of positive definite kernels into elementary products, *Doklady Akad. Nauk. SSSR*, **53** (#1) (1946); [3] The theory of self-adjoint extensions of half-bounded Hermitean operators and their applications, Part I, *Mat. Sbornik*, *20 (62)*: **3** (1947), 431–495; [4] Part II of the above article, *Mat. Sbornik*, *21 (63)*: **3** (1947), 366–404; [5] On Hermitean operators with directed functionals, *Sbornik prac. Inst. Mat. A. N. Ukr. SSR*, **10.**

KREIN, M. and M. A. KRASNOSELSKI — [1] Fundamental theorems on the extensions of Hermitean operators and some of their applications to the theory of orthogonal polynomials and the moment problem, *Uspekhi mat. Nauk*, **2** (Part 3) (1947).

KREIN, M., M. A. KRASNOSELSKI and D. P. MILMAN — [1] On the defect numbers of linear operators in Banach space and on some geometric questions, *Sbornik trudov Inst. A. N. Ukr. SSR*, **11** (1948).

LANDAU, E. — [1] Über einen Konvergenzsatz, *Göttinger Nachrichten*, (1907), 25–27.

LENGYEL, B. — [1] On the spectral theorem of self-adjoint operators, *Acta Sci. Math. Szeged*, **9** (1939), 174–186.

LENGYEL, B. A. and M. H. STONE — [1] Elementary proof of the spectral theorem, *Ann. of Math.*, **37** (1936), 853–864.

LEVITAN, B. M. — [1] *Almost periodic functions*, Moscow (1953); [2] *Entwicklung nach Eigenfunktionen*, Gostechizdat (1950).

LIFSCHITZ, M. S. — [1] An application of the theory of Hermitean operators to the generalized moment problem, *Doklady Akad. Nauk SSSR*, **44** (#1) (1944); [2] A class of linear operators in Hilbert space, *Mat. Sbornik*, **19** (61): **2** (1946), 239–260; [3] On the theory of isometric operators with equal deficiency numbers, *Doklady Akad. Nauk SSSR*, **58** (#1) (1947), 13–15; [4] On the theory of elementary divisors of Hermitean operators, *Doklady Akad. Nauk SSSR*, **60** (#1) (1948), 17–20.

LORCH, E. R. — [1] Functions of self-adjoint transformations in Hilbert space, *Acta Sci. Math. Szeged*, **7** (1934), 136–146; [2] On a calculus of operators in reflexive vector spaces, *Trans. Amer. Math. Soc.*, **45** (1939), 217–234; [3] The spectrum of linear transformations, *Trans. Amer. Math. Soc.*, **52** (1942), 238–248.

LÖWIG, H. — [1] Komplexe euklidischer Räume von beliebiger endlicher oder unendlicher Dimensionszahl, *Acta Sci. Math. Szeged*, **7** (1934), 1–33.

MIMURA, J. — [1] Über Funktionen von Funktionaloperatoren in einem Hilbertschen Raum, *Jap. J. Math.*, **13** (1936), 119–128.

MURRAY, F. J. — [1] *An introduction to linear transformations in Hilbert space*, Ann. of Math. Studies #4, Princeton (1941); [2] Linear transformations between Hilbert spaces and applications of this theory to linear partial differential equations, *Trans. Amer. Math. Soc.*, **37** (1935), 301–338.

MUSCHELISCHVILI, N. I. — [1] *Singular integral equations*, North Holland Pub. Co. (1953).

NAIMARK, M. A. - [1] Spectral functions of a symmetric operator, *Izv. A. N. SSSR*, **4** (#3) (1940), 277–318; [2] On self-adjoint extensions of the second kind of a symmetric operator, *Izv. A. N. SSSR*, **4** (#1) (1940), 53–104; [3] On the square of a closed symmetric operator, *Dokl. Akad. Nauk SSSR*, **26** (1940), 866-870; [4] On a representation of additive operator functions of sets, *Doklady Akad. Nauk SSSR*, **41** (1943), 373–375; [5] Extremal spectral functions of symmetric operators, *Izv. A. N. SSSR*, **11** (#4) (1947).

NAKANO, H. — [1] Zur Eigenwerttheorie normaler Operatoren, *Proc. Phys.-Math. Soc. Jap. III, Series 21* (1939), 315–339; [2] Über Abelsche Ringe von Projektionsoperatoren, *Proc. Phys.-Math. Soc. Jap. III, Series 21* (1939), 357–375; [3] Funktionen mehrerer hypermaximaler normaler Operatoren, *Proc. Phys.-Math. Soc. Jap. III, Series 21* (1939), 713–728; [4] Hypermaximalität normaler Operatoren, *Proc. Phys.-Math. Soc. Jap. III, Series 22* (1940), 259–264; [5] Unitärinvarianten hypermaximaler normaler Operatoren im Hilbertschen Raum, *Ann. of Math.*, **42** (1941), 657–664; [6] Über den Beweis des Stoneschen Satzes, *Ann. of Math.*, **42** (1941), 665–667; [7] Unitärinvarianten im allgemeinen euklidischen Raum, *Math. Ann.*, **118** (1941), 112–133.

NEUMANN, J. v. — [1] *Mathematische Grundlagen der Quantenmechanik*, Berlin (1932); [2] Zur Theorie der unbeschränkten Matrizen, *J. reine angew. Math.*, **161** (1929), 208–236; [3] Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, *Math. Ann.*, **102** (1929), 49–131; [4] Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren, *Math. Ann.*, **102** (1929), 370–427; [5] Über Funktionen von Funktionaloperatoren, *Ann. of Math.*, **32** (1931), 191–226; [6] Über adjungierte Funktionaloperatoren, *Ann. of Math.* **33** (1932), 294–310; [7] Über einen Satz von M. H. Stone, *Ann. of Math.*, **33**, (1932), 567–573; [8] Zur Operatorenmethode in der klassischen Mechanik, *Ann. of Math.*, **33** (1932), 587–648; [9] On normal operators, *Proc. Acad. Sci. U.S.A.*, **21** (1935), 366–369; [10] Charakterisierung des Spektrums eines Integraloperators, *Actualités Sci. et Ind.*, **229** (1935).

PALEY, R. E. A. C. and N. WIENER — [1] *Fourier transforms in the complex domain*, New York (1934).

PHILLIPS, R. S. — [1] On linear transformations, *Trans. Amer. Math. Soc.*, **48** (1940), 516–541.

PLANCHEREL, M. — [1] Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies, *Rend. Circ. Math. Palermo*, **30** (1910), 289–335; [2] Sur les formules de réciprocité du type de Fourier, *J. London Math. Soc.*, **8** (1933), 220–226.

PLESNER, A. I. — [1] Spectral theory of linear operators I, *Uspekhi mat. Nauk*, **9** (1941); [2] Über halbunitäre Operatoren, *Doklady Akad. Nauk SSSR*, **25** (1939), 710–712; [3] Über Funktionen eines maximalen Operators, *Doklady Akad. Nauk SSSR*, **23** (1939), 327–330; [4] Spectral analysis of maximal operators, *Doklady Akad. Nauk SSSR*, **22** (#5) (1939), 225–228.

PLESNER, A. I. and W. A. ROCHLIN — [1] Spectral theory of linear operators II, *Uspekhi mat. Nauk*, 1 (Part 1) (11) (1946).

POVZNER, A. J. — [1] On the method of directed functionals of M. G. Krein, *Zap. naučno-issl. Inst. mat. i mech. i Chark. mat. Obšč. Series 4*, **20** (1950).

REID, W. T. — [1] Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Math. J.*, **18** (1951), 41–56.

REDLICH, F. — [1] Spektraltheorie in nichtseparablen Räumen, *Math. Ann.*, **110** (1934), 342–356; [2] Störungstheorie der Spektralzerlegung I, *Math. Ann.* **113** (1936), 600–619; II, **113** (1936), 677–685; III, **116** (1939), 555–570; IV, **117** (1940), 356–382; V, **118** (1942), 462–484.

RIESZ, F. — [1] *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris (1931); [2] Untersuchungen über Systeme integrierbarer Funktionen, *Math. Ann.*, **69** (1910), 449–497; [3] Über die linearen Transformationen des komplexen Hilbertschen Raumes, *Acta Sci. Math. Szeged*, **5** (1930), 23–54; [4] Über Sätze von Stone und Bochner, *Acta Sci. Math. Szeged*, **6** (1933), 184–198; [5] Zur Theorie des Hilbertschen Raumes, *Acta Sci. Math. Szeged*, **7** (1934), 34–38; [6] Sur les fonctions des transformations hermitiennes dans l'espace de Hilbert, *Acta Sci. Math. Szeged*, **7** (1935), 147–159.

RIESZ, F. and E. R. LORCH — [1] The integral representation of unbounded self-adjoint transformations in Hilbert space, *Trans. Amer. Math. Soc.*, **39** (1936), 331–340.

RIESZ, F. and B. SZ.-NAGY — [1] Über Kontraktionen des Hilbertschen Raumes, *Acta Sci. Math. Szeged*, **10** (1943), 202–205; [2] *Functional analysis*, New York (1955).

SCHAUDER, J. — [1] Über lineare vollstetige Funktionaloperationen, *Studia Math.*, **2** (1930), 183–196.

SCHMIDT, E. — [1] Auflösung der allgemeinen linearen Integralgleichung, *Math. Ann.*, **64** (1907), 161–174; [2] Über die Auflösung linearer Gleichungen mit abzählbar unendlich vielen Unbekannten, *Rend Circ. Math. Palermo*, **25** (1908), 53–77.

SMIRNOV, V. I. — [1] Course of higher mathematics V, (1947).

SMITHIES, F. — [1] The Fredholm theory of integral equations, *Duke Math. J.*, **8** (1941), 107–130.

STONE, M. H. — [1] Linear transformations in Hilbert space, *Proc. Nat. Acad. Sci. U.S.A.*, **15** (1929), 198–200, 423–425; **16** (1930), 172–175; [2] On one-parameter unitary groups in Hilbert space, *Ann. of Math.*, **33** (1932), 643–648; [3] *Linear transformations in Hilbert space*, New York (1932).

STRAUS, A. W. — [1] Generalized resolvents of symmetric operators, *Izv. A. N. SSSR*, **18** (#1) (1954).

SUKHOMLINOV, G. — [1] Über Fortsetzung von linearen Funktionalen in linearen komplexen Räumen und linearen Quaternionenräumen, *Mat. Sbornik N. S.*, **3** (1938), 353–358.

SZ.-NAGY, B. — [1] Perturbations des transformations autoadjointes dans l'espace de Hilbert, *Commentarii Math. Helv.*, **19** (1946–47), 347–366; [2] On semi-groups of self-adjoint transformations in Hilbert space, *Proc. Nat. Acad. Sci. U.S.A.*, **24** (1938), 559–560; [3] Expansion theorems of Paley-Wiener type, *Duke Math. J.*, **14** (1947), 975–978; [4] On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math. Szeged*, **11** (1947), 152–157; [5] Perturbations des transformations linéaires fermées, *Acta Sci. Math. Szeged*, **14** (1951), 125–137; [6] *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Berlin (1942).

TAYLOR, A. E. — [1] Linear operations which depend analytically upon a parameter, *Ann. of Math.*, **39** (1938), 574–593; [2] Spectral theory of closed distributive operators, *Acta. Math.*, **84** (1950), 189–224.

TEICHMÜLLER, O. — [1] Operatoren im Wachschen Raum, *J. reine angew. Math.*, **174** (1935), 73–124.

TITCHMARCH, E. C. — [1] *Introduction to the theory of Fourier integrals*, Oxford Univ. Press (1937).

VIGIER, J. P. — [1] Étude sur les suites infinies d'opérateurs hermitiens, *Thesis, Geneva*, (1946).

VISSER, C. — [1] Note on linear operators, *Proc. Acad. Amsterdam*, **40** (1937), 270–272.

WECKEN, F. J. — [1] Zur Theorie linearer Operatoren, *Math. Ann.*, **110** (1935), 722–725; [2] Unitärinvarianten selbstadjungierter Operatoren, *Math. Ann.*, **116** (1939), 422–455.

WEYL, H. — [1] Über beschränkte quadratische Formen, deren Differenz vollstetig ist, *Rend. Circ. Math. Palermo*, **27** (1909), 373–392; [2] Über gewöhnliche Differentialgleichungen mit Singularitäten, *Math. Ann.*, **68** (1910), 220–269.

WHITTAKER, E. T. and G. N. WATSON — [1] *A course of modern analysis*, Cambridge (1952).

WINTNER, A. — [1] *Spektraltheorie unendlicher Matrizen*, Leipzig (1929).

ZAANEN, A. C. — [1] Über vollstetige symmetrische und symmetrisierbare Operatoren, *Nieuw Archief voor Wiskunde* (2), **22** (1948), 57–80; [2] On linear functional equations, *Nieuw Archief voor Wiskunde*, **22** (1948), 269–282.

ZYGMUND, A. — [1] *Trigonometrical series*, Warsaw (1935).

INDEX

Adjoint operator, 43, 79
Almost periodic function, 29, 132
Aperture, 69

Basis
  of a matrix representation, 99
  of a space, 19
Bessel's inequality, 15, 133
Bounded, 39

Cauchy-Bunyakovski inequality, 2
Characteristic function, 27
  of an interval, 27
Closed
  extension, 78
  operator, 78
  set, 4
  vector system, 16, 35
Closure
  of an operator, 78
  of a set, 4
  relation, 16
Compact, 46
Complement, orthogonal, 11
Complete
  orthonormal system, 17
  space, 4
Completely continuous operator, 56
Completeness, weak, 44
Component, 6
  with respect to a subspace, 10
Conjugation operator, 94
Continuity
  of a functional, 31
  of an operator, 31
Continuous spectrum, 91
Convergence
  in the mean, 23
  strong, 44, 61
  uniform, 61
  weak, 44, 61
Coordinate, 5

Dependent, linear, 1
Determinant, Gram, 12
Differential operator, 106
Dimension
  of a Hilbert space, 20
  of a linear manifold, 20
  of a linear space, 1
Direct sum, 11
Discrete spectrum, 91
Distance between two vectors, 4
Distribution function, 27

Eigenmanifold, 81
Eigenvalue, 81
  multiplicity, 81
Eigenvector, 81
Envelope, linear, 7
Equivalence
  of vector sets, 13
  unitary, 74
Euclidean space, 7
Extension, 31
  closed, 78
  minimal closed, 78

Fourier-Plancherel operator, 74, 112
Functionals, 30
  bilinear, 40, 42
  continuous, 31
  convex, 36
  linear, 32
  norm of, 32
Functions
  almost periodic, 29, 132
  characteristic, 27
  distribution, 27
  scalar, 30
  Tchebysheff-Hermite, 25
  Tchebysheff-Laguerre, 26
  vector, 30
Fundamental sequence, 4

Gram determinant, 12
Graph of an operator, 95

Hermite matrix, 54, 104
Hilbert
  relation, 92
  space, 5
Hilbert-Schmidt
  kernel, 54, 58
  operator, 54, 58

Independence, linear, 1
Integral operator with a Hilbert-Schmidt kernel, 54, 58
Invariant subspace, 81
Inverse operator, 30
Inversion formula of a singular integral, 114
Irreducible operator, 82
Isometric operator, 73, 87
  maximal, 87
Isomorphism
  of two Hilbert spaces, 20
  of two operators, 74

Jacobi matrix, 60

Kernel
  Hilbert-Schmidt, 54, 58
  of an integral operator, 54

Legendre polynomials, 25
l.i.m. (limes in medio), 23
Limit
  in the mean, 23
  point, 4
Linear
  combination, 1
  envelope, 7
  manifold, 7
  operator, 39
  space, 1
Linearly
  dependent system, 1
  independent system, 1
l^2-space, 5
l^p-space, 39

L^2-space, 21
L^2_σ-space, 27
L^p-space, 38

Manifold, linear, 7
Matrix
  generalized Jacobi, 60
  Hermitian, 54, 102
  Jacobi, 60
  symmetric, 52
Matrix representation of an operator, 50, 98
  basis of, 99
Maximal
  isometric operator, 87
  symmetric operator, 85
Metric space, 3
Metrizable, 2
Minimal closed extension, 78
Moment of a distribution function, 28

Neighborhood, 4
Norm
  of a bilinear functional, 40
  of a linear functional, 32
  of a linear operator, 39
  of a vector, 2
Normal operator, 43

Operator, 30
  adjoint, 43, 79
  closed, 78
  completely continuous, 56
  conjugation, 94
  differential, 106
  Fourier-Plancherel, 74, 112
  Hilbert-Schmidt, 54, 58
  inverse, 30
  irreducible, 82
  isometric, 73, 87
  linear, 39
  multiplication, 103
  norm of, 39
  normal, 43
  projection, 63
  real, 94
  self-adjoint, 43, 85

symmetric, 85
unitary, 72, 87
Orthogonal
complement, 11
sum, 11
vectors, 3
Orthonormal
basis, 19
sequence, 13
system, 13

Parseval relation, 16
Polynomials
Legendre, 25
Tchebysheff-Hermite, 25
Tchebysheff-Laguerre, 26
Product
of operators, 31
scalar, 2
Projection, 10, 66
Projection operator, 63
monotone sequences of, 68

Real operator, 94
Reduced
projection operator, 82
subspace, 82
Regular point, 88, 91
Resolvent, 91
Restriction, 81

Scalar product, 2
Self-adjoint operator, 43, 85
Separable, 5
length, 27
measurable functions, 27
measure, 27
Space
complete, 4
Euclidean, 7
linear, 1
metric, 3
metrized, 2
Spectrum
continuous, 91
discrete, 91
point, 91
Sphere in a Hilbert space, 4
Strong
compactness, 46
convergence, 44, 61
Subspace
invariant, 81
reduced, 82
Sum
direct, 11
orthogonal, 11
Symmetric
matrix, 52
operator, 85

Tchebysheff-Hermite
function, 25
polynomial, 25
Tchebysheff-Laguerre
function, 26
polynomial, 26
Theorem
of F. Riesz, 33
of H. Bohr, 134
Transformation, Fourier-Plancherel, 74
Triangle inequality, 3
Trigonometric system, 24

Uniform convergence of operators, 61
Unitary
equivalence, 74
operator, 72, 87

Weak
compactness, 46
completeness, 44
convergence, 44, 61
Weight, 28